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Cauchy problems related to integrable deformations of pseudo differential operators

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ABSTRACT

In this paper we discuss the solvability of two Cauchy problems in the pseudo differential operators. The first is associated with a set of pseudo differential operators of negative order, the prominent example being the set of strict integral operator parts of the different powers of a solution of the KP hierarchy. We show that it can be solved, provided the setting possesses a compatibility completeness. In such a setting all solutions of the KP hierarchy are obtained by dressing with the solution of the related Cauchy problem. The second Cauchy problem is slightly more general and links up with a set of pseudo differential operators of order zero or less. The key example here is the collection of integral operator parts of the different powers of a solution of the strict KP hierarchy. This system is solvable as soon as exponential and compatibility completeness holds. Also under these circumstances, all solutions of the strict KP hierarchy are obtained by dressing with the solution of the corresponding Cauchy problem.

1. Introduction

In [1], we discussed two types of deformations inside the algebra Psd of pseudo differential operators, depending on different decompositions of this algebra. The evolution of these deformations was formulated in Lax equations depending of the projection on one of the components of the decomposition. This Lax form of the system was shown to be equivalent to a set of zero curvature equations. This indicates already that there might be linear systems of which they are the compatibility relations.

Here we present two Cauchy problems in Psd. One that yields, for an appropriate choice of the operators involved, the zero curvature relations of the KP hierarchy and the other one, those of the strict KP hierarchy. The freedom one has in solving these systems consists of right multiplication with operators that are constant w.r.t. all the parameters involved. If Psd has a suitable specialization, like putting the parameters to zero in the formal power series setting, this enables you to gauge the solutions of the Cauchy problems. For the systems linked with the two hierarchies this freedom does not affect the solutions of the hierarchies that one can get in this way. As for the solvability of the Cauchy problems, we need compatibility completeness in the first case and besides that, we also need in the second case exponential completeness. Both properties hold in the formal power series context. Under these conditions all the solutions of the KP hierarchy and its strict version can be obtained from the solutions of the corresponding Cauchy problems.

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The content of the various sections is as follows: Section 1 recalls the necessary results for Psd from [1] and it shows the equivalence of the zero curvature equations for the complementary set of pseudo differential operators with the zero curvature equations from [1]. The next section formulates the two associated Cauchy problems, we analyze the freedom they allow and show when they can be solved. We end with the implications of the results for both hierarchies.

2. Integrable deformations in Psd

In this section we shortly recall the results needed from [1] about integrable deformations in the pseudo differential operators. We start with an algebra R over a field k of characteristic zero and a privileged k-linear derivation $\partial : R \mapsto R$.

Given R and ∂, one forms the algebra $R[\partial]$ of differential operators in ∂ with coefficients from R. It consists of k-linear endomorphisms of R of the form $\sum_{i=0}^{n} a_i\partial^i$, $a_i \in R$. If the powers of ∂ are not R-linear independent, then we decouple these relations and pass to the algebra $R[\xi]$ of formal expressions

$$\sum_{i=0}^{n} a_i\xi^i, \quad a_i \in R \text{ for all } i \geqslant 0.$$

Their addition and multiplication with scalars is done component wise; the product structure of $R[\xi]$ is given by the following rule:

$$\left(\sum_{i=0}^{n} a_i\xi^i\right)\left(\sum_{j=0}^{m} b_j\xi^i\right) = \sum_{\substack{0\leqslant i\leqslant n\\ 0\leqslant j\leqslant m}} \sum_{0\leqslant k\leqslant i} \binom{i}{k} a_i\partial^k(b_j)\xi^{i+j-k}. \tag{1}$$

Next one extends the algebra $R[\xi]$ by adding the inverses of all the powers of ξ and by allowing infinite sums of these negative powers. Thus one arrives at the algebra $\mathrm{Psd} = R[\xi, \xi^{-1})$ of all formal series

$$p = \sum_{j=-\infty}^{N} p_j\xi^j, \quad p_j \in R.$$

If one uses for each $n \in \mathbb{Z}$, the notation

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!},$$

then the product of two series $a = \sum_j a_j\xi^j$ and $b = \sum_i b_i\xi^i$ is similar to (1) and is given by

$$ab := \sum_j \sum_i \sum_{s=0}^{\infty} \binom{j}{s} a_j\partial^s(b_i)\xi^{i+j-s}.$$

Inside Psd we will make use of a number of decompositions. For $s \in \mathbb{Z}$, any pseudo differential operator $P = \sum_j p_j\xi^j \in \mathrm{Psd}$ can be split as

$$P = P_{\geqslant s} + P_{<s}, \quad \text{where } P_{\geqslant s} = \sum_{j\geqslant s} p_j\xi^j \quad \text{and} \quad P_{<s} = \sum_{j<s} p_j\xi^j. \tag{2}$$

For $s = 0$, this yields in particular the splitting of P in the differential operator part $P_{\geqslant 0}$ of P and its strict integral operator part $P_{<0}$. Similarly, we have

$$P = P_{>s} + P_{\leqslant s}, \quad \text{where } P_{>s} = \sum_{j>s} p_j\xi^j \quad \text{and} \quad P_{\leqslant s} = \sum_{j\leqslant s} p_j\xi^j. \tag{3}$$

For $s = 0$, this corresponds to writing P as the sum of its pure differential operator part $P_{>0}$ and its integral operator part $P_{\leqslant 0}$.

As any associative k-algebra, also Psd is w.r.t. the commutator a Lie algebra over k. From the multiplication rules in Psd follows that for $s = 0$ the two decompositions (2) and (3) yield two ways to split the Lie algebra Psd into the direct sum of two Lie subalgebras. The first being given by

$$\mathrm{Psd} = \{P \in \mathrm{Psd}, P = P_{\leqslant 0}\} \oplus \{P \in \mathrm{Psd}, P = P_{>0}\} := \mathrm{Psd}_{\leqslant 0} \oplus \mathrm{Psd}_{>0},$$

and the second one by

$$\mathrm{Psd} = \{P \in \mathrm{Psd}, P = P_{<0}\} \oplus \{P \in \mathrm{Psd}, P = P_{\geqslant 0}\} := \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geqslant 0}.$$

We write $\pi_{\geqslant 0}$ for the projection from Psd on $\mathrm{Psd}_{\geqslant 0}$ consisting of taking the differential operator part of an element in Psd. Similarly, one defines the projections of Psd on $\mathrm{Psd}_{\leqslant 0}$, $\mathrm{Psd}_{>0}$ and $\mathrm{Psd}_{<0}$, by $\pi_{\leqslant 0}$, $\pi_{>0}$ and $\pi_{<0}$, respectively.

To the Lie algebra $\mathrm{Psd}_{\leqslant 0}$ we associated the group

$$D(0) = \left\{ p_0 + \sum_{j<0} p_j\xi^j \,\middle|\, p_0 \in R^* \right\}$$