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Cohomological uniqueness of the Cauchy problem solutions for the Einstein–Maxwell equation

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ABSTRACT

In this paper we analyze the Cauchy problem for the Einstein–Maxwell equation in the case of non-characteristic initial hypersurface. To find the correct notions of characteristic and Cauchy data we introduce a complex, which we call the Einstein–Maxwell complex. Then the Cauchy problem acquires correctness in terms of an associated spectral sequence. We define a Cauchy data in such way that they allow us to reconstruct a cohomologously unique formal solution.

1. Introduction

Let M be a 4-dimensional manifold and let g be a metric on M of signature $(1, 3)$.

The *Einstein equation* is the system of 10 quasilinear 2-order partial differential equations on 10 independent components of the metric g on M, having the form (see, for example, [1])

$$\mathcal{R}(g) - \frac{1}{2}R(g)g + \Lambda g = \frac{8\pi k}{c^4}\mathcal{T} \tag{1}$$

where $\mathcal{R}(g)$ and $R(g)$ are respectively the Ricci tensor and the scalar curvature of the metric g, Λ is a cosmological constant, k is a gravitational constant, c is the velocity of light, and $\mathcal{T}$ is an energy–momentum tensor of matter.

Taking the trace of both sides of this equation, we get

$$R(g) = 4\Lambda - \frac{8\pi k}{c^4}\mathrm{tr}_g(\mathcal{T}).$$

Eq. (1) can be presented now in the equivalent form

$$\mathcal{R}(g) - \Lambda g = \frac{8\pi k}{c^4}\left(\mathcal{T} - \frac{1}{2}\mathrm{tr}_g(\mathcal{T})g\right). \tag{2}$$

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If $\mathcal{T} = 0$, then we have the *vacuum Einstein equation*:

$$\mathcal{R}(g) - \Lambda g = 0. \tag{3}$$

The *Maxwell equation* is the following system of 8 linear 1-order partial differential equations on 6 independent components of differential 2-form F on M (see, for example, [2])

$$\begin{cases} dF = 0, \\ \delta F - J = 0, \end{cases} \tag{4}$$

where $\delta = *d*$, $*$ is the Hodge $*$-operator, and J is *a current*, that is a differential 1-form on M such that

$$\delta J = 0. \tag{5}$$

A solution F of Eq. (4) is called an *electromagnetic tensor* or a *Faraday tensor*.

The *energy–momentum tensor of electromagnetic field F* has the form (see, [1])

$$\mathcal{T}(F)_{ij} = \frac{1}{4\pi}\left(-F_{im}F_{j\cdot}^{\ \cdot m} + \frac{1}{4}g_{ij}F_{mn}F^{mn}\right). \tag{6}$$

It is easy to check that

$$\mathrm{tr}_g\big(\mathcal{T}(F)\big) = 0. \tag{7}$$

Hence the Einstein equation in the presence of electromagnetic field F is the following

$$\mathcal{R}(g) - \Lambda\, g = \frac{8\pi k}{c^4}\mathcal{T}(F).$$

We see from the first equation of system (4) that there exists (at least locally) a differential 1-form φ on M such that $d\varphi = F$.

In terms of potential form φ Eq. (4) can be rewritten in the form

$$\delta d\varphi - J = 0. \tag{8}$$

By the *Einstein–Maxwell equation* we mean the following system of 2-order partial differential equations on components of metric g and differential 1-form φ

$$\begin{cases} \mathcal{R}(g) - \Lambda\, g - \dfrac{8\pi k}{c^4}\mathcal{T}(d\varphi) = 0, \\ \delta d\varphi - J = 0, \end{cases} \tag{9}$$

where $\mathcal{T}(d\varphi)$ is the energy–momentum tensor of electromagnetic field $F = d\varphi$.

The differential operator

$$\mathcal{E} : g \longmapsto \mathcal{R}(g) - \Lambda g$$

associated with Eq. (3), we call *Einstein operator* and the differential operator

$$\mathcal{EM} : \begin{pmatrix} g \\ \varphi \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{R}(g) - \Lambda\, g - \dfrac{8\pi k}{c^4}\mathcal{T}(d\varphi) \\ \delta d\varphi - J \end{pmatrix},$$

associated with Eq. (9), *Einstein–Maxwell operator*.

Let $N \subset M$ be a submanifold of codimension 1, g_0 a metric on M, φ_0 a differential 1-form on M, $u_0 = (g_0, \varphi_0)$, and $j^1_N u_0$ a 1-jet of u_0 on the submanifold N (see Section 6). Classically, the Cauchy problem for Eq. (9) consists of finding a pair $u = (g, \varphi)$, where g is a metric and φ is a differential 1-form on M, so that:

(1) $\mathcal{EM}(u) = 0$,
(2) $j^1_N u_0 = j^1_N u$.

In this case, the 1-jet $j^1_N u_0$ is said to be *Cauchy data*.

It is worth noting that the Cauchy problem for the Einstein–Maxwell equation is not correctly posed. There are at least two reasons for this.

The first one follows from the classical PDEs theory: it is easy to check that every submanifold $N \subset M$, $\mathrm{codim}\, N = 1$, is characteristic (see [3]).

The second one gives the geometric explanation of this phenomenon. Indeed, the Einstein–Maxwell equation satisfied to principle of naturality — it is invariant with respect to diffeomorphisms of the manifold M. Therefore any solution of the Cauchy problem can be transformed to a new one by a diffeomorphism preserving N and $j^1_N u_0$.

Note that exactly the same phenomenon holds for the vacuum Einstein equation, see [4,5]. To overcome the problem and construct formal solutions, we linearized the operator $\mathcal{E}$ on a solution g of the vacuum Einstein equation and include