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Dressing method for the vector sine-Gordon equation and its soliton interactions

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HIGHLIGHTS

- The explicit formulas for kink and breather solutions are derived.
- The method can be used to construct multi-soliton solutions.
- The soliton interactions are studied in detail.

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ABSTRACT

In this paper, we develop the dressing method to study the exact solutions for the vector sine-Gordon equation. The explicit formulas for one kink and one breather are derived. The method can be used to construct multi-soliton solutions. Two soliton interactions are also studied. The formulas for position shift of the kink and position and phase shifts of the breather are given. These quantities only depend on the pole positions of the dressing matrices.

1. Introduction

This paper is devoted to the study of an $O(n)$-invariant generalisation of the sine-Gordon equation

$$D_t\left(\frac{\vec{\alpha}_x}{\beta}\right)=\vec{\alpha},\qquad \beta^2+\langle\vec{\alpha},\vec{\alpha}\rangle=1, \tag{1}$$

where the dependent variable $\vec{\alpha}=(\alpha^1,\dots,\alpha^n)^T$ is n-dimensional real vector field and $\beta\in\mathbb{R}$. Here and in what follows the upper index T denotes the transpose of a vector or a matrix. We use the notation $\langle\cdot,\cdot\rangle$ for the Euclidean dot product of two vectors.

Eq. (1) first appeared in [1] was viewed as a reduction of the two-dimensional $O(n)$ nonlinear σ-model [2]. Its integrability properties were further studied afterwards. The Lax pairs were given in [3] and its Lagrangian formulation in [4]. Later, this equation reappeared in the study of connection between finite dimensional geometry, infinite dimensional geometry and integrable systems [5]. It was derived as the inverse flow of the vector modified Korteweg–de Vries equation

$$\vec{u}_\tau=\vec{u}_{xxx}+\frac{3}{2}\langle\vec{u},\vec{u}\rangle\vec{u}_x,\quad \vec{u}=\frac{\vec{\alpha}_x}{\beta}, \tag{2}$$

whose Hamiltonian, symplectic and hereditary recursion operators were naturally derived using the structure equation for the evolution of a curve embedded in an n-dimensional Riemannian manifold with constant curvature [6]. These have been recently re-derived in [7]. Besides, a partial classification of vector sine-Gordon equations using symmetry tests was done in [8].

Eq. (1) is a higher-dimensional generalisation of the well-known scalar sine-Gordon equation

$$\theta_{xt}=\sin\theta. \tag{3}$$

Indeed, it can be obtained by taking the dimension $n=1$ and letting $\beta=\cos\theta$ and $\alpha^1=\sin\theta$. The scalar sine-Gordon equation originates in differential geometry and has profound applications in physics and in life sciences (see recent review [9]). Vector generalisations of integrable equations have proved to be useful in applications [10]. They can be associated with symmetric spaces [11].

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The rational dressing method was originally proposed in [2,12] and developed in [13]. This method enables one to construct multi-soliton solutions and analyse soliton interactions in detail using basic knowledge of Linear Algebra. In this paper, we develop the dressing method for the vector sine-Gordon equation (1) and show that similar to the scalar sine-Gordon equation (3) there are two distinct types of solitons, namely kinks and breathers. One kink solution is a stationary wave propagating with a constant velocity. We show that a kink solution of the vector sine-Gordon equation can be obtained from a kink solution of (3) by setting $\vec{\alpha} = \mathbf{a}\sin\theta,\ \beta = \cos\theta$, where $\mathbf{a}$ is a constant unit length vector in $\mathbb{R}^n$. A general two kink solution of (1) cannot be obtained from solutions of (3), but it can be seen as a two kink solution of a vector sine-Gordon equation (1) with $n = 2$. One breather solution is a localised and periodically oscillating wave moving with a constant velocity. One breather solution of the general $O(n)$ invariant Eq. (1) can be obtained from a breather solution of (1) with $n = 2$ by an appropriate $O(n)$ rotation. Two breathers solution can be obtained from the corresponding solution of (1) with $n = 4$, etc. Surprisingly, the effects of interaction, such as the displacement and a phase shift (for breathers) are exactly the same as in the case of the scalar sine-Gordon equation (3) [14]. Such interaction properties are naturally valid for the vector modified Korteweg–de Vries equation (2). The detailed study of soliton interactions for (2) when $n = 2$ can be found in [15].

2. Dressing method for the vector sine-Gordon equation

In this section, we begin with the Lax representation of the vector sine-Gordon equation (1) given in [5], which is invariant under the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We then study the conditions for the dressing matrix (assumed to be rational in spectral parameter) with the same symmetries. The 1-soliton solutions of (1) correspond to the dressing matrix with only simple poles belonging to a single orbit of the reduction group. For one kink, it has two pure imaginary simple poles and for one breather, it has four complex simple poles. Using the dressing method, we explicitly derive one kink and one breather solutions starting with a trivial solution.

The vector sine-Gordon equation (1) is equivalent to the compatibility condition [5] $[\mathcal{L}, \mathcal{A}] = 0$ for two linear problems

$$\mathcal{L}\Psi = 0, \qquad \mathcal{A}\Psi = 0, \tag{4}$$

where

$$\mathcal{L} = D_x - \lambda J - U \quad \text{and} \quad \mathcal{A} = D_t + \lambda^{-1} V, \tag{5}$$

and

$$J = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ -1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 0_n \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & -\vec{\alpha}_x^T/\beta \\ \mathbf{0} & \vec{\alpha}_x/\beta & 0_n \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & \beta & \vec{\alpha}^T \\ -\beta & 0 & \mathbf{0}^T \\ -\vec{\alpha} & \mathbf{0} & 0_n \end{pmatrix}, \tag{6}$$

where $\mathbf{0}$ is n-dimensional zero column vector and 0_n is the $n \times n$ zero matrix. Without causing confusion, we sometimes simply write 0 instead.

The Lax operators $\mathcal{L}$ and $\mathcal{A}$ are invariant under the (reduction) group of automorphisms generated by the following three transformations: the first one is

$$\iota : \mathcal{L}(\lambda) \to -\mathcal{L}^{\dagger}(\lambda), \tag{7}$$

where $\mathcal{L}^{\dagger}(\lambda)$ is the adjoint operator defined by $\mathcal{L}^{\dagger}(\lambda) = -D_x - \lambda J^T - U^T$. The invariance under this transformation implies the matrices J and U are skew-symmetric. The second one is

$$r : \mathcal{L}(\lambda) \to \overline{\mathcal{L}(\bar{\lambda})}, \tag{8}$$

where $\overline{\mathcal{L}(\lambda)}$ means its complex conjugate. The invariance under this transformation reflects that the entries of matrices U and V are real. The last one is called Cartan involution

$$s : \mathcal{L}(\lambda) \to Q\mathcal{L}(-\lambda)Q, \tag{9}$$

where $Q = \mathrm{diag}(-1, 1, \dots, 1)$, which leads to the reduction to the symmetric space.

These three commuting transformations satisfy

$$\iota^2 = r^2 = s^2 = \mathrm{id}$$

and therefore generate the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Indeed, the operator $\mathcal{A}$ is also invariant under it, that is,

$$\iota(\mathcal{A}(\lambda)) = \mathcal{A}(\lambda), \qquad r(\mathcal{A}(\lambda)) = \mathcal{A}(\lambda), \qquad s(\mathcal{A}(\lambda)) = \mathcal{A}(\lambda). \tag{10}$$

Thus we say the Lax representation of (1) is invariant under the reduction group [13,16,17] $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

In what follows, we use the method of rational dressing [2,12,13] to construct new exact solutions of (1) starting from an exact solution $\vec{\alpha}_0,\ \beta_0$. Let us denote by U_0, V_0 the matrices U, V in which $\vec{\alpha},\ \beta$ are replaced by the exact solution $\vec{\alpha}_0,\ \beta_0$ of (1). The corresponding overdetermined linear system

$$\mathcal{L}_0\Psi_0 = (D_x - \lambda J - U_0)\Psi_0 = 0$$
$$\mathcal{A}_0\Psi_0 = (D_t + \lambda^{-1} V_0)\Psi_0 = 0 \tag{11}$$

has a fundamental solution $\Psi_0(\lambda, x, t)$ invariant under transformations (7)–(9). Following [2,12] we shall assume that the fundamental solution $\Psi(\lambda, x, t)$ for the new ("dressed") linear problems

$$\mathcal{L}\Psi = (D_x - \lambda J - U)\Psi = 0 \qquad \mathcal{A}\Psi = (D_t + \lambda^{-1} V)\Psi = 0 \tag{12}$$

is of the form

$$\Psi = \Phi(\lambda)\Psi_0, \qquad \det\Phi \neq 0, \tag{13}$$

where the dressing matrix $\Phi(\lambda)$ is assumed to be rational in the spectral parameter λ and to be invariant with respect to symmetries

$$\Phi(\lambda)^{-1} = \Phi^T(\lambda), \qquad \overline{\Phi(\bar{\lambda})} = \Phi(\lambda), \qquad Q\Phi(-\lambda)Q = \Phi(\lambda). \tag{14}$$

Conditions (14) guarantee that the corresponding Lax operators $\mathcal{L}$ and $\mathcal{A}$ are invariant with respect to transformations (7)–(9).

It follows from (11)–(13) that

$$\Phi(D_x - \lambda J - U_0)\Phi^{-1} = -\lambda J - U; \tag{15}$$

$$\Phi(D_t + \lambda^{-1} V_0)\Phi^{-1} = \lambda^{-1} V. \tag{16}$$

These equations enable us to specify the form of the dressing matrix Φ and construct the corresponding "dressed" solution $\vec{\alpha}, \beta$ of the vector sine-Gordon equation (1).

Let us consider the most trivial case when the dressing matrix Φ does not depend on the spectral parameter λ. In this case the dressing results in a point transformation ($O(n)$ rotation) of the initial solution $\vec{\alpha}_0$.

Proposition 1. *Assume that Φ is a λ independent dressing matrix for the vector sine-Gordon equation (1). If it is invariant with respect to symmetries (14), then*

$$\Phi = \pm \begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \Omega \end{pmatrix}, \tag{17}$$

where $\Omega \in O(n, \mathbb{R})$ is a constant (x, t-independent) matrix. The corresponding "dressed" solution is $\vec{\alpha} = \Omega\,\vec{\alpha}_0,\ \beta = \beta_0$.