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Jump bifurcations in some degenerate planar piecewise linear differential systems with three zones

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HIGHLIGHTS

- Borning of limit cycles from a continuum of homoclinic and heteroclinic connections.
- Bifurcation characterized by the birth of a limit cycle from a continuum of equilibria.
- The Morris–Lecar model for the activity of a single neuron activity is studied.

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ABSTRACT

We consider continuous piecewise-linear differential systems with three zones where the central one is degenerate, that is, the determinant of its linear part vanishes. By moving one parameter which is associated to the equilibrium position, we detect some new bifurcations exhibiting jump transitions both in the equilibrium location and in the appearance of limit cycles. In particular, we introduce the *scabbard bifurcation*, characterized by the birth of a limit cycle from a continuum of equilibrium points.

Some of the studied bifurcations are detected, after an appropriate choice of parameters, in a piecewise linear Morris–Lecar model for the activity of a single neuron activity, which is usually considered as a reduction of the celebrated Hodgkin–Huxley equations.

1. Introduction and statement of main results

The family of piecewise linear differential systems has become an important class of differential systems, due to its capability to model a large number of engineering problems, see [1–4] and references therein, as well as models from mathematical biology, see [5–8]. Despite of its seeming simplicity, there still are unsolved problems regarding stability and bifurcation issues.

In the case of planar systems with two linearity zones separated by a straight line, a lot of effort has been devoted to characterize the maximal number of limit cycles in the discontinuous setting [9–15], since the continuous case was already solved in [16], see also [17]. However, keeping the continuity of the vector field and dealing even with problems in low-dimensional phase spaces, the study of their dynamics is not completely done.

In this work, we want to clarify some bifurcation phenomena that can appear in planar continuous piecewise linear (CPWL) differential systems with three zones, without any special symmetry conditions but under a specific degeneracy. In particular, we consider the consequences of the vanishing of the determinant for the Jacobian matrix of the central zone. As will be shown, this hypothesis leads to a discontinuous behavior in the evolution of the equilibrium point with respect to the selected bifurcation parameter; this fact is rather counter-intuitive as long as the vector field depends continuously on such parameter.

Furthermore, regarding dynamic bifurcations, we reproduce some boundary equilibrium bifurcations leading to limit cycles. In particular, we find:

- an explosive generation of a limit cycle from a continuum of homoclinic and heteroclinic connections, similar to the one studied in [18];
- the generation of small limit cycles that grow linearly in amplitude with the bifurcation parameter, as in [17]; and
- we also encounter some specific bifurcations, as the introduced *scabbard bifurcation*, characterized by the birth of a limit cycle from a continuum of equilibrium points which, up to the best of our knowledge, has not been reported in the literature.

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We focus our attention on piecewise linear differential systems with three different regions separated by parallel straight lines, which can be assumed without loss of generality to be the lines $x = -1$ and $x = 1$, see [19]. Thus we have three regions of linearity, namely

$$S_L = \{(x, y) \in \mathbb{R}^2 : x < -1\}, \qquad S_C = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1\}$$

and

$$S_R = \{(x, y) \in \mathbb{R}^2 : x > 1\},$$

separated by the straight lines

$$\Sigma_\pm = \{(x, y) \in \mathbb{R}^2 : x = \pm 1\}.$$

Furthermore, it is rather usual for these systems to exhibit only one equilibrium point, whose position can be controlled by moving one parameter. This happens in particular when all the determinants of the involved linear parts are positive. Then, under these generic assumptions, see [19], and denoting with α the main bifurcation parameter, our CPWL systems can be written in the Liénard form

$$\begin{aligned} \dot{x} &= F(x) - y, \\ \dot{y} &= g(x) - \alpha, \end{aligned} \tag{1}$$

where the dot denotes derivatives with respect to a time variable τ,

$$F(x) = \begin{cases} t_R(x-1) + t_C, & \text{if } x \geq 1, \\ t_C x, & \text{if } |x| \leq 1, \\ t_L(x+1) - t_C, & \text{if } x \leq -1, \end{cases} \tag{2}$$

and

$$g(x) = \begin{cases} d_R(x-1) + d_C, & \text{if } x \geq 1, \\ d_C x, & \text{if } |x| \leq 1, \\ d_L(x+1) - d_C, & \text{if } x \leq -1. \end{cases}$$

Note that the three matrices ruling the dynamics in system (1) are

$$\begin{bmatrix} t_L & -1 \\ d_L & 0 \end{bmatrix}, \qquad \begin{bmatrix} t_C & -1 \\ d_C & 0 \end{bmatrix}, \qquad \begin{bmatrix} t_R & -1 \\ d_R & 0 \end{bmatrix},$$

where t_Z and d_Z with $Z \in \{L, C, R\}$ denote the trace and determinant in each linear zone.

Note that the above formulation includes as particular cases the following ones. If $t_C = t_L$ and $d_C = d_L$ then we have a system with only two different linearity zones, thoroughly analyzed in [16]. If $t_R = t_L$, $d_R = d_L$ and $\alpha = 0$, then we have a symmetric system with three different linearity zones, thoroughly analyzed in [20]. Non-symmetric systems were considered in [21,22]. A simpler case included into the previous formulation was studied in [23], where authors consider the non-generic situation $d_R > 0$, $t_R = 0$ and $d_C > 0$.

Remark 1. Note that CPWL systems are Lipschitz and so they satisfy the standard results on existence and uniqueness of solution as well as their continuous dependence respect to initial conditions and parameters. In fact, the solutions are functions of class $\mathcal{C}^1$ and we emphasize that several classical results of the qualitative theory of planar differential systems, see [24], and in particular Poincaré-Bendixson's and Dulac's theorems can be adequately extended to cover these CPWL systems.

Our initial assumption on the uniqueness of equilibrium point required the determinants in all the three zones to be positive. In this paper we will consider instead a degenerate situation by assuming that the determinant in the central zone vanishes, that is, $d_C = 0$, keeping the original assumptions $d_L, d_R > 0$. This setting arises in a natural way when one wants to analyze certain Petri nets, see for instance [25]. By considering the second equation in (1), equilibrium points should be located at points $(x, y) = (\bar{x}, \bar{y})$, where $\bar{x}$ is any solution of $g(x) = \alpha$ and $\bar{y} = F(\bar{x})$, being now

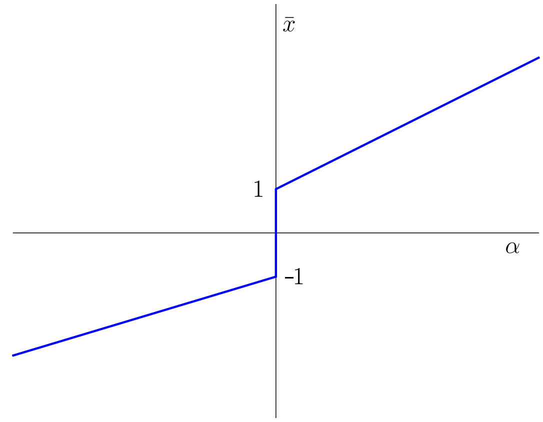

Fig. 1. The graph of $\bar{x}$ in terms of parameter α.

$$g(x) = \begin{cases} d_R(x-1), & \text{if } x \geq 1, \\ 0, & \text{if } |x| \leq 1, \\ d_L(x+1), & \text{if } x \leq -1. \end{cases} \tag{3}$$

Thus, regarding the equilibrium solutions of system (1)–(3), we can state the first consequence of the above assumptions. The proof of this first result is straightforward and will be omitted.

Lemma 1. *The following statements hold for system (1)–(3).*

(a) *For $\alpha < 0$ the system has only one equilibrium point, which is in the left zone, namely at*

$$e_L = (\bar{x}_L, \bar{y}_L) = \left(-1 + \frac{\alpha}{d_L}, \frac{\alpha t_L}{d_L} - t_C\right).$$

(b) *For $\alpha = 0$ the system has a continuum of non-isolated equilibrium points, which are in the central zone, namely at every point of the segment*

$$E_C = \{(\bar{x}, \bar{y}) : -1 \leq \bar{x} \leq 1, y = t_C \bar{x}\}.$$

(c) *For $\alpha > 0$ the system has only one equilibrium point, which is in the right zone, namely at*

$$e_R = (\bar{x}_R, \bar{y}_R) = \left(1 + \frac{\alpha}{d_R}, \frac{\alpha t_R}{d_R} + t_C\right).$$

It should be noticed that, when α passes through the critical value $\alpha = 0$, the system exhibits a jump transition in the equilibrium position from the left zone to the right one, see Fig. 1. This transition can be also associated to a change in the stability and topological type of the equilibrium, depending on the values of the linear invariants t_Z, d_Z of the external zones, where $Z \in \{L, R\}$. Also, as it will be later shown, the transition could be accompanied with the appearance or disappearance of a limit cycle. In this sense, regarding the traces t_L, t_C, t_R of each zone, we know from Bendixson theory that they all cannot have the same sign to allow the existence of limit cycles.

Since the number of different possibilities is high, here we only consider the cases where $t_L < 0$ and $t_R > 0$, so that the transition is associated to passing from one stable equilibrium point to one unstable one. Once restricted to such case, we must distinguish the different signs of the trace t_C and the different possible dynamics in the external zones (focus or node). To halve the length of our study, we will impose that the dynamics in the right zone is of focus type, that is, we will assume $t_R^2 - 4d_R < 0$.

Whenever we have a focus dynamics in an external zone, it is convenient to introduce some crucial parameters, namely

$$\gamma_Z = \frac{t_Z}{2\omega_Z}, \quad \text{where } \omega_Z = \sqrt{d_Z - \frac{t_Z^2}{4}} \tag{4}$$