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# Inertial effects on thin-film wave structures with imposed surface shear on an inclined plane

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## HIGHLIGHTS

- A thin-film model of flow on an inclined plane with imposed surface shear.
- Investigate effects of inertia on possible wave structures.
- Stabilisation of capillary ripples due to surface shear.
- Smoothing of shock structures due to inertial effects.
- Appearance of new wave structures due to inertia under certain flow conditions.

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## ABSTRACT

This study provides an extended approach to the mathematical simulation of thin-film flow on a flat inclined plane relevant to flows subject to high surface shear. Motivated by modelling thin-film structures within an industrial context, wave structures are investigated for flows with moderate inertial effects and small film depth aspect ratio $\varepsilon$. Approximations are made assuming a Reynolds number, $\mathrm{Re} \sim \mathcal{O}\left(\varepsilon^{-1}\right)$ and depth-averaging used to simplify the governing Navier–Stokes equations. A parallel Stokes flow is expected in the absence of any wave disturbance and a generalisation for the flow is based on a local quadratic profile. This approach provides a more general system which includes inertial effects and is solved numerically. Flow structures are compared with studies for Stokes flow in the limit of negligible inertial effects. Both two-tier and three-tier wave disturbances are used to study film profile evolution. A parametric study is provided for wave disturbances with increasing film Reynolds number. An evaluation of standing wave and transient film profiles is undertaken and identifies new profiles not previously predicted when inertial effects are neglected.

## 1. Introduction

Oil films on the internal surfaces of an aero-engine bearing chamber are a primary mechanism in removing heat from the chamber as oil is continuously collected, externally cooled and recycled. In a generic bearing chamber, the oil film is typically driven by a strong shearing airflow, associated with high-speed rotating parts within the chamber. Inertial effects, relevant to high-speed applications, have been included within a recent two-dimensional film flow formulation [1]. Using this approach, leading nonlinear inertial effects are retained to analyse more general flow patterns and to provide comparison with existing thin-film asymptotic studies. Importantly the study aims to investigate the effect of inertia on some typical thin-film solutions, obtain greater insight into existing thin-film solutions and seek more general film solutions.

Lubrication theory is typically used to model general thin-film flows, based on the ratio of film thickness to a typical geometric length scale being sufficiently small and the effects of inertia negligible; i.e. taking the limit as the film Reynolds number tends to zero. However, for films subject to a high surface shear, the inertial effects at leading order may have a substantial impact in modifying high-speed film profiles, providing a general smoothing effect on the film profiles, extending the range of existing film solutions and forming possible new film profiles. Recent studies [1] within a cylindrical geometry demonstrate that modified wave structures are predicted on including film inertial effects into a film lubrication formulation.

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Numerous studies involving flow on an inclined plane range from near-horizontal flows [2] to vertical falling films [3–5]. Previous studies, for example [6,7], for shear driven flow on an inclined plane describe wave structures depending on their initial profiles as well as uniform upstream and downstream boundary conditions. Initial profiles were chosen to be of a two-tier configuration, with either the upstream or downstream film height the higher of the two. Additionally various three-tier configurations may exist in which the central section may be the highest or lowest of three heights or a successive increase or decrease in the three heights, as shown later in Figs. 4 and 5.

It is common to make thin-film approximations of the Navier–Stokes equations, using the ratio $\varepsilon$ of film thickness and typical length scale of disturbances of the flow; typically the long wave equation or the integral boundary layer approach. Often used for describing falling films, the long wave equation, and other similar models, are derived using a perturbation expansion method for the stream function, determined to $\mathcal{O}(\varepsilon)$, and then substituted into the surface kinematic boundary condition. An early publication using this method by Benney [8] gives a single evolution equation for film thickness, however this has been criticised as predicting unrealistic wave profiles as the wave amplitude increases [5], as well as leading to finite-time singularities [9,10].

The integral boundary layer model takes the full Navier–Stokes equations and applies a thin-film assumption, leading to neglecting several terms. A local velocity profile, assumed parabolic by many, including [4,5,10], is then substituted into this thin-film model. The Kármán–Polhausen depth-averaging technique can be applied, integrating throughout the thickness of the film to reduce the dimensionality of the system, achieving what is commonly referred to, in literature, as the Shkadov model. The Shkadov model, too, has its criticisms. It does not predict the Hopf bifurcation, necessary to foresee the formation of periodic waves on uniform thickness film flow on inclined planes [5]. Ruyer-Quil et al. [6] claim that fluid films in a moving frame of reference flowing down an inclined plane are unstable against waves when their thickness becomes larger than a specified threshold value $h_{Nc} = (3\mathrm{R}_c)^{\frac{1}{3}}$, $\mathrm{R}_c$ being a critical Reynolds number. The critical Reynolds number is given by Cheng and Chang [11] as $\mathrm{R}_c = \cot\theta$ who claim that periodic forcing applied at the inlet leads to disturbances propagating downstream whilst growing in amplitude—this happens at Reynolds numbers larger than critical [12]. An example of sinusoidal waves at the inlet which have evolved into much larger amplitude solitary pulses is given in [4]. Ruyer-Quil et al. [13] also claim that there are limitations to the Shkadov model deriving from the lack of freedom in the description of the hydrodynamic fields, due to the nature of the depth-averaging technique. Notably, the simplification has the benefit of reduced computational cost on reducing the dimensionality of the Navier–Stokes equation.

Furthermore, thin-film flows having free surface shock structures are known to form capillary ripples at the front of the shock. In some cases, breakdown of the numerical results can occur in the simulation of Shkadov models due to over-prediction of capillary wave amplitudes, resulting in very thin film thickness. The accuracy of the Shkadov model in predicting capillary ripples is shown to decrease with increasing Reynolds number (see Figures 4 and 5 of [13]), with larger than actual capillary waves being predicted. Indeed, an essential assumption of this model concerns the number of degrees of freedom of polynomials used to approximate the crosswise distribution of the streamwise velocity. A study by Malamataris et al. [14] observes that a self-similar parabolic profile provides a good approximation of the velocity field in regions where deviations are small, but also concludes that there is a change of behaviour around the region in front of a wave hump. Computed velocity profiles with backflow are depicted in Figure 7 of [14], with potential inflexion points which could indicate that a cubic profile may be a more suitable approximation in this region. This is in agreement with work by Samanta et al. [9], who conclude that in a nonlinear regime, the backflow phenomenon is shown to be intensified by a no-slip condition enforced between the fluid and the plane. Backflow in the capillary region is investigated experimentally by Dietze et al. [15]. It should also be noted that Malamataris et al. comment on a parabolic profile, which is the exact solution of uniform film flow, as making the problem analytically tractable. Studies by Prokopiou et al. [12] and Yu et al. [16] extend the Shkadov model in order to allow for higher order description of film profiles relevant at higher Reynolds numbers. Referred to as the second-order boundary layer model, this retains terms of $\mathcal{O}(\varepsilon^2)$, and hence the modified model includes additional viscous terms, tangential and normal stress conditions and pressure variations across the film.

An investigation of travelling wave solutions on an inclined plane by Benilov [17] describes a depth-averaged approach based on the Stokes equations, i.e. neglecting inertial effects. It is suggested in this study that the model can be extended with a term accounting for surface tension.

The above mentioned literature predominantly investigate gravity driven film flow. Several other studies take similar approaches but with the imposition of surface shear. Kay et al. [1] study two-dimensional thin-film rimming flow within a fixed cylindrical geometry subject to surface shear. Extending previous leading-order thin-film models, inertial effects are included relevant to very high surface shear and correspondingly high-speed film flow applications. Using a hydraulic model approach enabled leading non-linear inertial effects to be retained. Kay et al. use a local quadratic velocity profile for its consistency with lubrication theory, but also extended to a cubic velocity profile, where the extra profile parameter permits modelling of wall roughness effects. Important to this study, the effects of inertia on some typical rimming-flow solutions provide new and greater insight into existing film solutions.

Samanta [18] investigates the mechanism of instability in shear imposed flow on an inclined plane. The geometry of the problem studied is very similar to the model we propose but with disturbances generated at the inlet which evolve into periodic travelling waves. There are also significant differences in the order of scaling used—the dimensionless governing equations include diffusive terms which, as will later be shown, are disregarded in our formulation due to the nature of the thin-film approximations. Another important comparison is the magnitude of the Reynolds number which differs from what we term the reduced Reynolds number by $\varepsilon$. The avenue of exploration in [18] is predominantly concerning the linear instability threshold and recovering the critical Reynolds and Froude numbers for the shear-imposed falling film. Although we will not be conducting any formal stability analysis, Section 5 contains some discussion about the evolution of perturbations in order to characterise waves structures.

A numerical study based on the Stokes equations using a boundary element approach was given by Shuaib et al. [7] to determine the different patterns of possible wave structures which can be observed on a thin film flowing on an inclined plane when subject to a constant surface shear, and categorises conditions for different types of film profiles. Studies by Bertozzi et al. [19] proposed that there are several different types of shock waves that connect films of different heights on an inclined plane when the flow is driven by surface shear and gravity. Solutions with more than one type of shock wave were also reported, dependent on the initial configuration of the flow. Bertozzi reports three different types of wave profiles, namely compressive shocks, undercompressive shocks and rarefaction waves. In order to obtain