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Modulational instability in nonlinear nonlocal equations of regularized long wave type

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HIGHLIGHTS

- A modulational instability index for BBM type equations.
- A modulational instability index for regularized Boussinesq type equations.
- Modulational instability of supercritical small wave trains of the BBM equation.
- Modulational stability of small wave trains of the regularized Boussinesq equation.
- An instability diagram for fractional dispersion.

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ABSTRACT

We study the stability and instability of periodic traveling waves in the vicinity of the origin in the spectral plane, for equations of Benjamin–Bona–Mahony (BBM) and regularized Boussinesq types permitting nonlocal dispersion. We extend recent results for equations of Korteweg–de Vries type and derive modulational instability indices as functions of the wave number of the underlying wave. We show that a sufficiently small, periodic traveling wave of the BBM equation is spectrally unstable to long wavelength perturbations if the wave number is greater than a critical value and a sufficiently small, periodic traveling wave of the regularized Boussinesq equation is stable to square integrable perturbations.

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1. Introduction

We study the stability and instability of periodic traveling waves for some classes of nonlinear dispersive equations, in particular, equations of Benjamin–Bona–Mahony (BBM) type

$$u_t + \mathcal{M}(u+u^2)_x = 0 \tag{1.1}$$

and regularized Boussinesq type

$$u_{tt} - \mathcal{M}^2(u+u^2)_{xx} = 0. \tag{1.2}$$

Here, $t \in \mathbb{R}$ is typically proportional to elapsed time and $x \in \mathbb{R}$ is the spatial variable in the primary direction of wave propagation; $u = u(x,t)$ is real valued, representing the wave profile or a velocity. Throughout we express partial differentiation either by a subscript or using the symbol ∂. Moreover $\mathcal{M}$ is a Fourier multiplier, defined via its symbol as

$$\widehat{\mathcal{M}f}(k) = m(k)\widehat{f}(k)$$

and characterizing dispersion in the linear limit. Note that

$$m(k) = \text{the phase speed} \quad \text{and} \quad (km(k))' = \text{the group speed}. \tag{1.3}$$

Throughout the prime means ordinary differentiation.

Assumption 1.1. We assume that

(M1) m is real valued and twice continuously differentiable,
(M2) m is even and, without loss of generality, $m(0) = 1$,
(M3) $C_1|k|^\alpha < m(k) < C_2|k|^\alpha$ for $|k| \gg 1$ for some $\alpha \geqslant -1$ and $C_1, C_2 > 0$,
(M4) $m(k) \neq m(nk)$ for all $k > 0$ and $n = 2, 3, \ldots$.

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Assumption (M1) ensures that the spectra of the associated linearized operators depend in the C^1 manner on the (long wavelength) perturbation parameter; here we are not interested in achieving a minimal regularity requirement. Assumption (M2) is to break that (1.1), or (1.2), is invariant under spatial translations. Assumption (M3) ensures that periodic traveling waves of (1.1) or (1.2) are smooth, among others. Assumption (M4) rules out the resonance between the fundamental mode and a higher harmonic.

The present treatment works *mutatis mutandis* for a broad class of nonlinearities. Here we assume for simplicity the quadratic power-law nonlinearity. Incidentally it is characteristic of numerous wave phenomena.

In the case of $\mathcal{M} = (1-\partial_x^2)^{-1}$, note that (1.1) reduces to the BBM equation

$$u_t - u_{xxt} + u_x + (u^2)_x = 0, \tag{1.4}$$

which was proposed in [1], as an alternative to the Korteweg–de Vries (KdV) equation

$$u_t + u_x + u_{xxx} + (u^2)_x = 0, \tag{1.5}$$

to model long waves of small but finite amplitude in a channel of water. In the case of $\mathcal{M}^2 = (1-\partial_x^2)^{-1}$, moreover, (1.2) reduces to the regularized Boussinesq equation

$$u_{tt} = u_{xxtt} + u_{xx} + (u^2)_{xx}. \tag{1.6}$$

It does not explicitly appear in the work of Boussinesq. But (280) in [2], for instance, after several "higher order terms" drop out, becomes equivalent to what Whitham derived in [3, Section 13.11]. Under the assumption that $u_t + u_x$ is small (which implies right running waves), one may, in turn, derive (1.6), or the singular Boussinesq equation

$$u_{tt} = u_{xxxx} + u_{xx} + (u^2)_{xx}. \tag{1.6$'$}$$

Moreover (1.6) finds relevance in other physical situations such as nonlinear waves in lattices; see [4], for instance. The phase speed of a plane wave solution with the wave number k of the linear part of (1.6) is (see (1.3))

$$\sqrt{\frac{1}{1+k^2}} = 1 - \frac{1}{2}k^2 + O(k^4) \quad \text{for } k \ll 1,$$

and it agrees up to the second order with the phase speed $\sqrt{1-k^2}$ for (1.6′) when k is small. Hence (1.6) and (1.6′) are equivalent for long waves. But (1.6) is preferable over (1.6′) for short and intermediately long waves. As a matter of fact, the initial value problem associated with the linear part of (1.6′) is ill-posed, because a plane wave solution with $k > 1$ grows unboundedly, whereas arbitrary initial data lead to short time existence for (1.6). Note that (1.2) factorizes into two sets of (1.1) — one moving to the left and the other to the right.

Related to (1.1) and (1.2) are equations of KdV type

$$u_t = (\mathcal{M}u + u^2)_x. \tag{1.7}$$

Note that (1.1), (1.2) and (1.7) share the dispersion relation in common, but their nonlinearities are different. They are *nonlocal* unless m, or m^{-1} in the case of (1.1) and (1.2), is a polynomial in ik. Examples include the Benjamin–Ono equation, for which $m(k) = |k|$ in (1.7), and the intermediate long wave equation, for which $m(k) = k \coth k$ in (1.7). Another example, which Whitham proposed in [3] to argue for wave breaking in shallow water, corresponds to $m(k) = \sqrt{\tanh k/k}$ in (1.7); see [5], for instance, for details.

By a traveling wave of (1.1), (1.2) or (1.7), we mean a solution which progresses at a constant speed without change of form. For a broad class of dispersion symbols, periodic traveling waves with small amplitude may be attained from a perturbative argument, for instance, a Lyapunov–Schmidt reduction; see Appendix A for details. We are interested in their stability and instability in the vicinity of the origin in the spectral plane. Physically, it amounts to long wavelength perturbations or slow modulations of the underlying wave.

Whitham in [6,7] (see also [3]) developed a formal asymptotic approach to study the effects of slow modulations in nonlinear dispersive waves. Since then, there has been considerable interest in the mathematical community in rigorously justifying predictions from Whitham's modulation theory. Recently in [8–11] (see also [12]), in particular, long wavelength perturbations were carried out analytically for (1.7) and for a class of Hamiltonian systems permitting nonlocal dispersion, for which Evans function techniques and other ODE methods may not be applicable. Specifically, modulational instability indices were derived either with the help of variational structure (see [8]) or using asymptotic expansions of the solution (see [9–11]).

Theorem 1.2 (*[10,11]*). *Under Assumption 1.1, a $2\pi/k$-periodic traveling wave of (1.7) with sufficiently small amplitude is spectrally unstable with respect to long wavelength perturbations if*

$$ind_{KdV}(k) := \frac{i_1(k)i_2^-(k)i_{KdV}(k)}{i_3^-(k)} < 0, \tag{1.8}$$

where

$$\begin{aligned} i_1(k) &= (km(k))'', \\ i_2^-(k) &= (km(k))' - 1, \\ i_3^-(k) &= m(k) - m(2k) \end{aligned} \tag{1.9}$$

and

$$i_{KdV}(k) = 2i_3^-(k) + i_2^-(k). \tag{1.10}$$

Otherwise, it is stable to square integrable perturbations in the vicinity of the origin in the spectral plane.

Here we take matters further and derive modulational instability indices for (1.1) and (1.2).

Theorem 1.3 (*Modulational Instability Index for (1.1)*). *Under Assumption 1.1, a sufficiently small, $2\pi/k$-periodic traveling wave of (1.1) is spectrally unstable to long wavelength perturbations if*

$$ind_{BBM}(k) := \frac{i_1(k)i_2^-(k)i_{BBM}(k)}{i_3^-(k)} < 0, \tag{1.11}$$

where i_1, i_2^-, i_3^- are in (1.9) and

$$i_{BBM}(k) = 2i_3^-(k) + m(2k)i_2^-(k). \tag{1.12}$$

Otherwise, it is stable to square integrable perturbations in the vicinity of the origin in the spectral plane.

Theorem 1.4 (*Modulational Instability Index for (1.2)*). *Under Assumption 1.1, a sufficiently small, $2\pi/k$-periodic traveling wave of (1.2) is spectrally unstable to long wavelength perturbations if*

$$ind_{Bnesq}(k) := \frac{i_1(k)i_2^-(k)i_2^+(k)i_{Bnesq}(k)}{i_3^-(k)i_3^+(k)} < 0, \tag{1.13}$$

where i_1, i_2^-, i_3^- are in (1.9),

$$\begin{aligned} i_2^+(k) &= (km(k))' + 1, \\ i_3^+(k) &= m(k) + m(2k) \end{aligned} \tag{1.14}$$

and

$$i_{Bnesq}(k) = 2i_3^-(k)i_3^+(k) + m^2(2k)i_2^-(k)i_2^+(k). \tag{1.15}$$