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# First-order aggregation models with alignment

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## HIGHLIGHTS

- We study a first-order aggregation model that includes attraction, repulsion and alignment interactions.
- We require momentum conservation in the first-order model to insure uniqueness of solutions.
- We show how the model can be rigorously obtained as a zero-inertia limit of a second-order kinetic model.
- A numerical scheme that parallels the analysis is developed and implemented; flocking behaviour is demonstrated.

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## ABSTRACT

We include alignment interactions in a well-studied first-order attractive–repulsive macroscopic model for aggregation. The distinctive feature of the extended model is that the equation that specifies the velocity in terms of the population density, becomes *implicit*, and can have non-unique solutions. We investigate the well-posedness of the model and show rigorously how it can be obtained as a macroscopic limit of a second-order kinetic equation. We work within the space of probability measures with compact support and use mass transportation ideas and the characteristic method as essential tools in the analysis. A discretization procedure that parallels the analysis is formulated and implemented numerically in one and two dimensions.



## 1. Introduction

The literature on self-organizing behaviour or swarming has grown dramatically over the last years. A variety of mathematical models has been proposed, which have origin in biological applications (e.g., self-collective behaviour seen in species such as fish, birds or insects [1]), as well as in social sciences and engineering (e.g., opinion formation [2], social networks [3], robotics and space missions [4]). The main aspect is the modelling of the social interactions between the members of a group; due to such inter-individual interactions, self-organization may occur in a physical space (insect swarms, fish schools, robots) or, more abstractly, in an opinion space.

One approach in modelling aggregation is to consider individuals/organisms as point particles and design either an ordinary differential equation (ODE) or a discrete-time equation to model their evolution. Another is to formulate a partial differential equation (PDE) that governs the time evolution of the population density field. These two approaches result from the various descriptions that one can take in modelling aggregation behaviour: particle-based/microscopic or continuum/macroscopic. We refer to [5] for a recent review of aggregation models, where in particular, it is shown how microscopic models can be related to macroscopic ones via kinetic theory.

Three types of social interactions have been commonly considered in the literature on mathematical aggregations: attraction, repulsion, and alignment. Consequently, aggregation models can be distinguished in terms of which of these interactions are being accounted for. Some models consider only a subset of these interactions (just attraction and repulsion [6–9] or just alignment [10,11]), others account for all three of them. Models of the latter type are typically referred to as "three-zone" models, as each particular interaction type acts at different ranges (repulsion

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acts at short distances, while alignment and attraction are present at intermediate and long ranges, respectively). This class of models has had many successful applications in biological and sociological modelling [12–14].

Aggregation models (discrete or continuous) may also differ in how the velocity field is determined. There are second-order/dynamic models, typically in the form of the Newton's second law, where a differential equation for the evolution of the velocity is being provided [8,10], and first-order/kinematic models where the velocity is prescribed in terms of the spatial configuration [6,15,7]. The aim of the present paper is to extend, by including alignment interactions, a first-order continuum model for aggregation that attracted a high amount of interest in recent literature [6,16–20]. Below we introduce the extended model and its derivation, then point out the fundamental issues that arise with such an extension, and how we address these issues in the present paper.

Consider the following continuum model for the evolution of the macroscopic density function $\rho(t, x)$ in $\mathbb{R}^d$:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \qquad \rho\big|_{t=0} = \rho_0(x), \tag{1.1a}$$

$$\Phi(t,x)u(t,x) = \int_{\mathbb{R}^d} \phi(|x-y|)\rho(t,y)u(t,y)\,\mathrm{d}y - \nabla_x K * \rho(t,x), \tag{1.1b}$$

where $\phi$ is an *influence* function that controls the alignment interactions, $\Phi = \phi*\rho$, and $K$ is an attractive–repulsive interaction potential. The asterisk denotes spatial convolution. Hence, the model consists in an active transport equation for the density $\rho$, with velocity field $u$ defined by (1.1b). The coefficient $\Phi(t, x)$ in the left-hand-side of (1.1b) has the interpretation of the total influence received at location $x$ and time $t$ from the rest of the group. The right-hand-side of (1.1b) has two terms: the first models alignment, and the second models attraction and repulsion, as detailed below.

Alignment is modelled through an averaging mechanism that allows individuals to adjust their velocities relative to the velocities of the others. Specifically, the velocity at location $x$ and time $t$ is assumed to depend non-locally on the velocities $u(t, y)$ at locations $y$ within the alignment interaction range set by the support of the influence function $\phi$. The first-term in the right-hand-side of (1.1b) captures this averaging process, with weights/interaction strengths given by $\phi(|x - y|)$, assumed to depend only on the relative distance between locations $x$ and $y$.

Attraction and repulsion are modelled by the convolution of the gradient of the interaction potential with the population density. In brief, individuals are assumed to repel each other at short ranges, to create a comfort zone around them, but attract each other once they distance themselves too far apart. Eq. (1.1a) with the velocity field given *solely* by this term, i.e., $u(t, x) = -\nabla_x K * \rho(t, x)$, constitutes the aggregation model that has been referred to above and which the present paper generalizes. A variety of issues has been investigated during the last decade for this explicit attractive–repulsive aggregation model, including the well-posedness of solutions [16,17,21,22], the long-time behaviour of solutions [6,23,18,19,9,20], and the derivation of the continuum model as a mean-field limit [24].

We point out an essential feature of Eq. (1.1b), which is that it is an *implicit* equation in $u$, and can have *non-unique* solutions (e.g., due to translational invariance, one can add an arbitrary function of $t$ to any solution of (1.1b), and obtain a different solution). This is a key challenge brought up by the inclusion of alignment interactions in the explicit aggregation model from [6]. Addressing this challenge is one of the major goals of the present paper. The non-uniqueness of solutions to models of type (1.1) has been noted in [14], but no resolution was offered. To our best knowledge, the present paper is the first systematic study of a *first-order* continuum model for aggregation that includes *both* attractive/repulsive and alignment interactions.

The origin of the macroscopic model (1.1) can be traced back to the following second-order discrete model derived from Newton's second law. Suppose there are $N$ particles in $\mathbb{R}^d$, whose positions and velocities denoted by $x_i$ and $v_i$, respectively ($i = 1, \ldots, N$), evolve according to the following system of ODE's:

$$\frac{dx_i}{dt} = v_i, \tag{1.2a}$$

$$\epsilon\frac{dv_i}{dt} = \frac{1}{N}\sum_{j\neq i}\phi(|x_j-x_i|)(v_j-v_i) - \frac{1}{N}\sum_{j\neq i}\nabla_{x_i}K(x_i-x_j), \tag{1.2b}$$

for $i = 1, \ldots, N$. Here, it has been assumed that all particles have the same mass $m_i = \epsilon$. The functions $\phi$ and $K$ have similar meaning as in (1.1).

Without the attractive–repulsive term modelled by the second term in the right-hand-side of (1.2b), model (1.2) represents the celebrated model of Cucker and Smale [10]. It is well-known that for certain influence functions $\phi$, the Cucker–Smale model successfully captures the unconditional *flocking* phenomenon, where individuals align their velocities into a certain asymptotic direction [25,26,11]. Comprehensive literature also exists on second-order attractive–repulsive models without alignment [8,27,5]. Second-order models with *both* alignment and attraction/repulsion have also been studied [28–31], though in not as much depth and detail as the models with the two sets of forces considered separately.

The passage from the second-order discrete model (1.2) to the first-order macroscopic model (1.1) is done in two steps. First, for each fixed $\epsilon > 0$, one can take the limit $N \to \infty$ in (1.2) and reach, by BBGKY hierarchies or mean field limits (see e.g., [25] and the review in [5]), the following kinetic equation for the density $f(t, x, v)$ at position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$:

$$\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon = \frac{1}{\epsilon}\nabla_v\cdot(F[f_\epsilon]f_\epsilon), \qquad f_\epsilon\big|_{t=0} = f_0(x,v), \tag{1.3a}$$

$$F[f_\epsilon] = \int_{\mathbb{R}^{2d}}\phi(|x-y|)(v-v^*)f_\epsilon(t,y,v^*)\,\mathrm{d}y\,\mathrm{d}v^* + \int_{\mathbb{R}^{2d}}\nabla_x K(x-y)f_\epsilon(t,y,v^*)dydv^*. \tag{1.3b}$$

Rigorous derivations of mean field limits starting from particle systems is a classical subject, comprising an extensive body of works. Some of the most recent works include the mean field limit of the Cucker–Smale model [26,11], as well as extensions of these results to include general aggregation models of the form (1.2) [30].

The second step in passing from (1.2) to (1.1) is to send $\epsilon \to 0$ in the kinetic equation (1.3) and derive (1.1) as a hydrodynamic limit. The rigorous treatment of this limit constitutes in fact a major component of the present work. Passage from kinetic to macroscopic equations is also a vast topic, extensively studied for instance in the context of hydrodynamic limits of the nonlinear Boltzmann equations. It is beyond the scope of this introduction to give a detailed account of this well-established research area, we simply refer here to a recent review paper [32] and the references therein.

As indicated above, a major issue that arises when one considers the first-order model (1.1) is the *non-uniqueness* of solutions to (1.1b). In the present paper we resolve this non-uniqueness issue for the case when the interaction potential is *symmetric* about the origin, i.e., it satisfies

$$K(x) = K(-x), \quad \text{for all } x \in \mathbb{R}^d. \tag{1.4}$$

For symmetric potentials, the ODE system (1.2) and the kinetic equation (1.3) conserve the linear momentum: $\sum_{i=1}^N v_i$ and