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Periodic motions of fluid particles induced by a prescribed vortex path in a circular domain

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HIGHLIGHTS

- A two-dimensional ideal fluid inside a circular domain under the action of a prescribed stirring protocol.
- The motion of advected particles follows a Hamiltonian system.
- The vortex induces a singularity on the angular variable.
- An infinite number of periodic solutions are found.

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ABSTRACT

By means of a generalized version of the Poincaré–Birkhoff theorem, we prove the existence and multiplicity of periodic solutions for a Hamiltonian system modeling the evolution of advected particles in a two-dimensional ideal fluid inside a circular domain and under the action of a point vortex with prescribed periodic trajectory.

1. Introduction and main result

We consider the motion of a two-dimensional ideal fluid in a circular domain of radius $R > 0$ subjected to the action of a moving point vortex whose position, denoted as $z(t)$, is a prescribed T-periodic function of time. This model plays an important role in Fluid Mechanics as an idealized model of the stirring of a fluid inside a cylindrical tank by an agitator. A fundamental reference for this problem is the seminal paper [1], where the concept of *chaotic advection* was coined. Following the classical Lagrangian representation, the mathematical model under consideration is the planar system

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i}\left(\frac{|z(t)|^2 - R^2}{(\zeta - z(t))(\zeta\bar{z}(t) - R^2)}\right), \tag{1}$$

where the complex variable ζ represents the evolution on the position of a fluid particle induced by the so-called *stirring protocol* $z(t)$. System (1) is a T-periodically forced planar system with Hamiltonian structure, where the stream function

$$\Psi(t,\zeta) = \frac{\Gamma}{2\pi}\ln\left|\frac{\zeta - z(t)}{\bar{z}(t)\zeta - R^2}\right|$$

plays the role of the Hamiltonian.

The main contribution of Aref in [1] was to show that the flow may experience regular or chaotic regimes depending on the particular stirring protocol. For instance, system (1) is integrable if $z(t)$ is constant or $z(t) = z_0\exp(i\Omega t)$ but it is chaotic if $z(t)$ is piecewise constant (blinking protocol in the related literature). A naive way to measure the influence of the ideas presented in [1] is to note the more than a thousand citations of this inspiring paper to date. Aref's blinking protocol is piecewise integrable and the theory of linked twist maps permits a good analytical study of the underlying dynamics (see for instance [2,3]). More recently, other

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strategies of stirring have been studied, for instance the figure-eight or the epitrochoidal protocol [4], but only from a numerical point of view. Our contribution in this paper is to prove that both regular and chaotic regimes share a common dynamical feature, namely the existence of an infinite number of periodic solutions labeled by the number of revolutions around the vortex in the course of a period.

To be precise, let us fix $z : \mathbb{R} \to \mathbb{C}$ a T-periodic function such that $|z(t)| < R$ for all t. For a periodic solution ζ of (1) with period kT, the *winding number* of ζ is defined as

$$\mathrm{rot}_{kT}(\zeta) = \frac{1}{2\pi i}\int_0^{kT} \frac{d(\zeta(t) - z(t))}{\zeta(t) - z(t)}$$

and provides the number of revolutions of $\zeta(t)$ around the vortex point $z(t)$ in the time interval $[0, kT]$. We proceed to state our main result.

Theorem 1.1. *Let $z : \mathbb{R} \to \mathbb{C}$ be a T-periodic function of class C^1, such that $|z(t)| < R$ for all t. Then, for every integer $k \geq 1$, system (1) has infinitely many kT-periodic solutions lying in the disk $\mathcal{B}_R(0)$. More precisely, for every integer $k \geq 1$, there exists an integer j_k^* such that, for every integer $j \geq j_k^*$, system (1) has two kT-periodic solutions $\zeta_{k,j}^{(1)}(t), \zeta_{k,j}^{(2)}(t)$ such that, for $i = 1, 2$,*

$$\|\zeta_{k,j}^{(i)}\|_\infty \leq R \quad \text{and} \quad \mathrm{rot}_{kT}(\zeta_{k,j}^{(i)}) = j. \tag{2}$$

Moreover, for every $k \geq 1, j \geq j_k^*$ and $i = 1, 2$,

$$\lim_{j\to+\infty} |\zeta_{k,j}^{(i)}(t) - z(t)| = 0, \quad \text{uniformly in } t \in [0, kT]. \tag{3}$$

In particular, for $k = 1$, we find that (1) has infinitely many T-periodic solutions. For $k > 1$, we find subharmonic solutions of order k (i.e., kT-periodic solutions which are not lT-periodic for any $l = 1, \dots, k-1$) provided that j and k are relatively prime integers; we remark that in this case it is also possible to show that $\zeta_{k,j}^{(1)}(t), \zeta_{k,j}^{(2)}(t)$ are not in the same periodicity class (namely, $\zeta_{k,j}^{(1)}(\cdot) \not\equiv \zeta_{k,j}^{(2)}(\cdot + lT)$ for every integer $l = 1, \dots, k-1$).

It is worth pointing out that the regularity condition on the stirring protocol plays an important role. In fact, Theorem 1.1 is not true for a discontinuous $z(t)$ (e.g. the blinking protocol), because condition (3) would imply unphysical discontinuous particle trajectories. The existence and multiplicity of periodic solutions for a general protocol, as well as their stability properties, remains as an open problem. We will come back to this issue in the final section.

The rest of the paper is divided in three section. In Section 2 the Poincaré section is defined, whereas Section 3 contains the proof of Theorem 1.1 by an application of a generalized version of the Poincaré–Birkhoff Theorem. The paper is concluded by Section 4 with a discussion on the physical meaning of the presented results and some other remarks.

2. Definition of the Poincaré section

For our purposes, it is convenient to write system (1) as

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i}\left(\frac{1}{\zeta - z(t)} - \frac{1}{\zeta - \frac{R^2}{|z(t)|^2}z(t)}\right). \tag{4}$$

In this form, the first term on the right models the action of the vortex whereas the second term corresponds to the wall influence on the flow. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ and setting $\zeta = (x, y), z(t) = (a(t), b(t))$, we can rewrite system (4) in real notation as

$$\begin{cases} \dot{x} = \dfrac{\Gamma}{2\pi}\left(-\dfrac{y - b(t)}{|\zeta - z(t)|^2} + \dfrac{y - \frac{R^2}{|z(t)|^2}b(t)}{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\ \dot{y} = \dfrac{\Gamma}{2\pi}\left(\dfrac{x - a(t)}{|\zeta - z(t)|^2} - \dfrac{x - \frac{R^2}{|z(t)|^2}a(t)}{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right), \end{cases}$$
$$\zeta = (x, y) \in \mathbb{R}^2. \tag{5}$$

Let $\mathcal{B}_R \subset \mathbb{R}^2$ be the closed disk centered at the origin with radius R. First, we recall a well known property of system (5).

Lemma 2.1. *Let $\zeta : J \to \mathbb{R}^2$ be a solution of (5), with $J \subset \mathbb{R}$ its maximal interval of definition. If $|\zeta(t_0)| \leq R$ for some $t_0 \in J$, then $|\zeta(t)| \leq R$ for every $t \in J$, that is to say, the disk $\mathcal{B}_R$ is invariant for the flow associated to (5).*

Proof. Since $\mathcal{B}_R = \{(x, y) \in \mathbb{R}^2 \mid V(x, y) \leq R^2\}$ for $V(x, y) = x^2 + y^2$, by the standard result of flow-invariant sets, it is enough to prove that

$$\langle Z(t, x, y)|\nabla V(x, y)\rangle = 0, \quad \text{for every } t \in [0, T],\ x^2 + y^2 = R^2,$$

where $Z(t, x, y)$ denotes the vector field of the differential system (5). With simple computations, we find indeed

$$\begin{aligned}\langle Z(t, x, y)|\nabla V(x, y)\rangle &= \frac{1}{2}\Big(X(t, x, y)x + Y(t, x, y)y\Big) \\ &= \frac{\Gamma}{\pi}\Big(b(t)x - a(t)y\Big)\left(\frac{\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2 - \frac{R^2}{|z(t)|^2}|\zeta - z(t)|^2}{|\zeta - z(t)|^2\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\ &= \frac{\Gamma}{\pi}\Big(b(t)x - a(t)y\Big)\left(\frac{\left(1 - \frac{R^2}{|z(t)|^2}\right)\left(|\zeta|^2 - \frac{R^2}{|z(t)|^2}|z(t)|^2\right)}{|\zeta - z(t)|^2\left|\zeta - \frac{R^2}{|z(t)|^2}z(t)\right|^2}\right) \\ &= 0. \quad \square\end{aligned}$$

From now on, we will study solutions to system (5) belonging to the invariant disk $\mathcal{B}_R$; accordingly, the singularity of the vector field at $\zeta = \frac{R^2}{|z(t)|^2}z(t)$ (for which $|\zeta| > R$) will not play any role. On the contrary, we will take advantage of the singularity at $\zeta = z(t)$. To this aim, it is useful to introduce the change of variable

$$\eta = \zeta - z(t)$$

and set $\eta = (u, v)$, so that system (5) is transformed into

$$\begin{cases} \dot{u} = \dfrac{\Gamma}{2\pi}\left(-\dfrac{v}{|\eta|^2} + \dfrac{v + b(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|\eta + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot{a}(t) \\ \dot{v} = \dfrac{\Gamma}{2\pi}\left(\dfrac{u}{|\eta|^2} - \dfrac{u + a(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)}{\left|\eta + z(t)\left(1 - \frac{R^2}{|z(t)|^2}\right)\right|^2}\right) - \dot{b}(t), \end{cases}$$
$$\eta = (u, v) \in \mathbb{R}^2. \tag{6}$$

In the following, given $\eta_0 \neq 0$, we will denote by $\eta(\cdot; \eta_0)$ the unique solution of (6) satisfying the initial condition $\eta(0) = \eta_0$.

Lemma 2.2. *There exists $r > 0$ such that, if $0 < |\eta_0| \leq r$, then the solution $\eta(\cdot; \eta_0)$ exists on $\mathbb{R}$ and satisfies $|\eta(t; \eta_0) + z(t)| \leq R$, for every $t \in \mathbb{R}$.*