Journal of Geometry and Physics 80 (2014) 1–14

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Normal forms for sub-Lorentzian metrics supported on Engel type distributions

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ARTICLE INFO

Article history:
Received 18 December 2012
Received in revised form 4 November 2013
Accepted 17 February 2014
Available online 24 February 2014

Keywords:
Sub-Lorentzian manifolds
Geodesics
Reachable sets
Geometric optimality
Engel distributions

ABSTRACT

We construct normal forms for Lorentzian metrics on Engel distributions under the assumption that abnormal curves are timelike future directed Hamiltonian geodesics. Then we indicate some cases in which the abnormal timelike future directed curve initiating at the origin is geometrically optimal. We also give certain estimates for reachable sets from a point.

1. Introduction

1.1. Preliminaries

In the series of papers [1–3] we studied (germs of) contact sub-Lorentzian structures on $\mathbb{R}^3$. In turn, in the series [4–6] some classes of non-contact sub-Lorentzian structures on $\mathbb{R}^3$ were studied (in all cases the underlying distribution is of rank 2). The next reasonable step is to study sub-Lorentzian structures again supported by rank 2 distributions but on $\mathbb{R}^n$, $n \geq 4$. In this paper we begin studies in this direction, namely we examine the simplest such case, i.e. one supported by the so-called Engel distribution. Before giving precise definition we will first present basis notions and facts from the sub-Lorentzian geometry that will be needed to state the results.

For all details and proofs the reader is referred to [7] (and to other papers by the author; see also [8,9]). Let M be a smooth manifold, and let H be a smooth distribution on M of constant rank. For a point $q \in M$ and an integer i let us define H_q^i to be the linear subspace in T_qM generated by all vectors of the form $[X_1, [X_2, \ldots, [X_{k-1}, X_k] \ldots]](q)$, where $X_1, \ldots, X_k$ are smooth (local) sections of H defined near q, and $k \leq i$. We say that H is *bracket generating* if for every $q \in M$ there exists a positive integer $i = i(q)$ such that $H_q^i = T_qM$. Now, by *a sub-Lorentzian structure (or metric)* on M we mean a pair (H, g) made up of a smooth bracket generating distribution H of constant rank and a smooth Lorentzian metric on H. A triple (M, H, g) is called *a sub-Lorentzian manifold.*

Up to the end of this subsection we fix a sub-Lorentzian manifold (M, H, g). A vector $v \in H_q$ is called *timelike* if $g(v, v) < 0$, is called *nonspacelike* if $g(v, v) \leq 0$ and $v \neq 0$, is *null* if $g(v, v) = 0$ and $v \neq 0$, finally is *spacelike* if $g(v, v) > 0$ or $v = 0$. By *a time orientation* of (H, g) we mean a continuous timelike vector field on M. Suppose that X is a time orientation of (M, H, g). Then a nonspacelike $v \in H_q$ is said to be *future directed* if $g(v, X(q)) < 0$, and is *past directed* if $g(v, X(q)) > 0$. An absolutely continuous curve $\gamma : [a, b] \longrightarrow M$ is called *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)}$ a.e. on $[a, b]$. A horizontal curve is nonspacelike (resp. timelike, null, nonspacelike future directed etc.) if so is $\dot{\gamma}(t)$ a.e.

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http://dx.doi.org/10.1016/j.geomphys.2014.02.006
0393-0440/

Below we will need a notion of Hamiltonian geodesics. Let $\mathcal{H} : T^*M \longrightarrow \mathbb{R}$ be the so-called*geodesic (or metric) Hamiltonian* associated with our structure (H, g). A global definition of $\mathcal{H}$ is given for instance in [7]. Locally $\mathcal{H}$ looks as follows. Take an orthonormal basis $X_0, \ldots, X_k$ for H defined on an open set $U \subset M$, where X_0 is timelike. Then the restriction of $\mathcal{H}$ to T^*U is given by $\mathcal{H}(q,p) = -\frac{1}{2}\langle p, X_0(q)\rangle^2 + \frac{1}{2}\sum_{j=1}^{k}\langle p, X_j(q)\rangle^2$. Denote by $\overrightarrow{\mathcal{H}}$ the Hamiltonian vector field corresponding to the function $\mathcal{H}$. A horizontal curve is called *a Hamiltonian geodesic* if it can be represented in the form $\gamma(t) = \pi \circ \lambda(t)$, where $\dot{\lambda} = \overrightarrow{\mathcal{H}}$ and $\pi : T^*M \longrightarrow M$ is the canonical projection. $\lambda(t)$ is called a Hamiltonian lift of $\gamma(t)$. It is immediate from the very definition that if $\gamma : [a, b] \longrightarrow M$ is a Hamiltonian geodesic and $\dot{\gamma}(t_0)$ is a nonspacelike (resp. timelike, null, nonspacelike future directed etc.) vector, then so is $\dot{\gamma}(t)$ for every $t \in [a, b]$.

Before going further, it seems sensible to clarify why we use the word 'geodesic'. So, first of all, if $\gamma : [a, b] \longrightarrow M$ is a nonspacelike curve then we define its *sub-Lorentzian length* by formula

$$L(\gamma) = \int_a^b \sqrt{-g(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

Next, for an open subset $U \subset M$ and any pair of points $q_1, q_2 \in U$, denote by $\Omega^{nspc}_{q_1,q_2}(U)$ the set of all nonspacelike future directed curves contained in U and joining q_1 to q_2. Now we say that a nonspacelike future directed curve $\gamma : [a, b] \longrightarrow U$ is a *maximizing U-geodesic* or simply a *U-maximizer* if

$$L(\gamma) = \max\left\{L(\eta) : \ \eta \in \Omega^{nspc}_{\gamma(a),\gamma(b)}(U)\right\}.$$

By a *U-geodesic* we mean a curve in U whose every sufficiently small subarc is a U-maximizer (such an approach follows the ideas elaborated in the Lorentzian case — see e.g. [10,11] or [12]). It turns out [7] that for every nonspacelike Hamiltonian geodesic $\gamma : [a, b] \longrightarrow M$ and for every $t \in (a, b)$ there exists a neighbourhood U of $\gamma(t)$ such that $U \cap \gamma$ is a U-maximizer. Note that in the Lorentzian (or Riemannian) geometry every geodesic is Hamiltonian. It is known that in the sub-Lorentzian (or sub-Riemannian) geometry there are maximizers (minimizers) that are not Hamiltonian geodesics — see e.g. [2] and Remark 1.1 below for examples in the sub-Lorentzian case (and [13,14] for the sub-Riemannian situation).

Denote by Φ_t the (local) flow of the field $\overrightarrow{\mathcal{H}}$. For a fixed point $q_0 \in M$ let us define $\mathcal{D}_{q_0}$ to be the set of all $\lambda \in T^*_{q_0}M$ such that the curve $t \longrightarrow \Phi_t(\lambda)$ is defined on the whole interval [0, 1]. $\mathcal{D}_{q_o}$ is an open subset in $T^*_{q_0}M$. Now we define *the exponential mapping* with the pole at q_0

$$\exp_{q_0} : \mathcal{D}_{q_0} \longrightarrow M, \qquad \exp_{q_0}(\lambda) = \pi \circ \Phi_1(\lambda).$$

Using properties of Hamiltonian equations it is easy to see that the Hamiltonian geodesic with initial conditions (q_0, λ) can be written as $\gamma(t) = \exp_{q_0}(t\lambda)$. It can also be observed that if $\gamma(t)$ is a Hamiltonian geodesic with a Hamiltonian lift $\lambda(t) = \Phi_t(\lambda)$ then, from the definition of the geodesic Hamiltonian (see [7] for more details), it follows that for any $v \in H_{\gamma(t)}$ we have

$$g(\dot{\gamma}(t), v) = \langle \Phi_t(\lambda), v\rangle. \tag{1.1}$$

At the end let us recall the notion of abnormal curves (cf. e.g. [13]). So an absolutely continuous curve $\lambda : [a, b] \longrightarrow T^*M$ is called an abnormal biextremal if $\lambda([a, b]) \subset H^{\perp}$, λ never intersects the zero section, and moreover $\Omega_{\lambda(t)}(\dot{\lambda}(t), \zeta) = 0$ for almost every $t \in [a, b]$ and every $\zeta \in T_{\lambda(t)}H^{\perp}$; here $H^{\perp}$ is the annihilator of H, and Ω denotes the restriction to $H^{\perp}$ of the standard symplectic form on T^*M. A horizontal curve $\gamma : [a, b] \longrightarrow M$ is said to be *abnormal* if there exists an abnormal biextremal $\lambda : [a, b] \longrightarrow T^*M$ such that $\gamma = \pi \circ \lambda$.

Throughout the paper we will use the following abbreviations: "t." for "timelike", "nspc." for "nonspacelike", and "f.d." for "future directed". Moreover, unless otherwise stated, we assume all curves and vectors to be horizontal. Thus e.g. a t.f.d. curve is a horizontal curve whose tangent is t.f.d. a.e.

1.2. Statement of the results

Let H be a *rank* 2 distribution of constant rank on a 4 -dimensional manifold M. We say that H is an *Engel (or Engel type) distribution* if H^2 is of constant rank 3, and H^3 is of constant rank 4, i.e. $H^3 = TM$. The remarkable property of Engel distributions is the fact that they are topologically stable, see e.g. [15] (note that apart from Engel case, the only stable distributions are rank 1 distributions, and also contact and pseudo-contact distributions). On the other hand, if one slightly perturbs any given rank 2 distribution on a 4-manifold it becomes Engel on an open and dense subset. All this gives rise to the importance of Engel distributions. But Engel distributions are important also because of another reason, namely they appear in applications. For instance our flat case (see example below) serves as a model for a motion of a car with a single trailer (cf. e.g. [16]).

Using for instance [13] one makes sure that if H is an Engel distribution on M then through each point $q \in M$ there passes exactly one unparameterized abnormal curve. Moreover the abnormal curves are all (at least locally) trajectories of a single smooth vector field.

Let H be an Engel type distribution and let g be a Lorentzian metric on H. A couple (H, g) is called an *Engel sub-Lorentzian structure (or metric)* if the abnormal curves for H are timelike. If moreover the abnormal curves are, possibly after reparameterization, t.f.d. Hamiltonian geodesics then (H, g) will be called *Engel sub-Lorentzian structure of Hamiltonian type*.