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Transparency of strong gravitational waves

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ABSTRACT

This paper studies diagonal spacetime metrics. It is shown that the overdetermined Einstein vacuum equations are compatible if one Killing vector exists. The stability of plane gravitational waves of the Robinson type is studied. This stability problem bares a fantastic mathematical resemblance to the stability of the Schwarzschild black hole studied by Regge and Wheeler. Just like for the Schwarzschild black hole, the Robinson gravitational waves are proven to be stable with respect to small perturbations. We conjecture that a bigger class of vacuum solutions are stable, among which are all gravitational solitons. Moreover, the stability analysis reveals a surprising fact: a wave barrier will be transparent to the Robinson waves, which therefore passes through the barrier freely. This is a hint of integrability of the $1+2$ vacuum Einstein equations for diagonal metrics.

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1. Introduction

In the theory of relativity, the Einstein–Hilbert action is

$$S = \frac{1}{2}\int R\sqrt{-g}d^4x \tag{1}$$

where R is the scalar curvature of the spacetime metric $g_{\mu\nu}$, g is the determinant of $g_{\mu\nu}$ and the integration is performed over the four-dimensional spacetime. Varying the Einstein–Hilbert action (1) with respect to the inverse metric $g^{\mu\nu}$ gives Einstein's vacuum equations,

$$R_{\mu\nu} = 0 \tag{2}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor. Einstein's vacuum equations determine the evolution of the spacetime metric $g_{\mu\nu}$ in empty space.

This paper focuses on *diagonal* spacetime metrics. These are metrics that can be written in the form

$$g_{\mu\nu} = (H_\mu)^2\delta_{\mu\nu} \tag{3}$$

where $\delta_{\mu\nu}$ is the Kronecker delta. Here and in the rest of this paper, Einstein's summation convention is *not* used. In matrix form, the diagonal metric is

$$g_{\mu\nu} = \begin{bmatrix} (H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{bmatrix}. \tag{4}$$

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It is convenient not to worry about the sign of the metric. Instead, one may restore the proper metric signature $(-+++)$ by substituting $H_0 \to iH_0$.

It is a well known result that *every* metric $g_{\mu\nu}$ may be diagonalized at *any* given event of spacetime (e.g. by using Riemann normal coordinates) [1]. Nevertheless, this is a local result, which holds globally only for very specific spacetime metrics. This means that the class of metrics that can be casted into the diagonal form (3) *globally* should be expected to have *unique features*. It is important to keep in mind that the *diagonality* of the metric *is not an invariant property*. In other words, some non-diagonal metrics $g_{\mu\nu}$ may be transformed to the diagonal form (3) by a proper choice of coordinates.

The metric (3) describes a wide range of physical phenomena. In particular, it includes the Schwarzschild black hole [2], the Kasner metric [3], the Friedmann–Robertson–Walker model of cosmology [4], the Milne model of cosmology [4], a certain class of single-polarized plane gravitational waves [4] and special cases of gravitational solitons [5].

The goal of this work is to study the system of vacuum Einstein Eqs. (2) for diagonal metrics (3). The Christoffel symbols and energy–momentum tensor were previously derived in [6]. In this paper we extend this result by deriving the Einstein equations for diagonal metrics, study their compatibility and analyze their implications on gravitational waves.

Section 2 includes a derivation of the Einstein equations in the case of the diagonal metric (3). A convenient form for analyzing the equations is obtained. Section 3 shows that if at least one Killing vector exists, Einstein's equations for diagonal metric are compatible. In Section 4, plane gravitational waves are studied. A simple criteria for asymptotic flatness and compatibility of the field equations for plane waves are derived. One of the most famous examples of such plane waves is the Bondi–Pirani–Robinson (BPR) waves [7]. In Section 5 it is proven that such waves are stable with respect to diagonal perturbations that depend on $1+2$ coordinates.

As a concrete example, in Section 6 a BPR wave with soliton-like properties is studied. The emitted (perturbation) wave is shown to travel through the BPR wave *without any reflection* and *independently of the amplitude of the BPR wave*. The latter implies that a *strong (BPR) gravitational wave would be transparent* to the perturbation wave. The only remnant of the collision is a phase shift which depends on the angle between the two waves. These properties, which are typically exhibited by solitons, *suggest that the* $1+2$ *vacuum Einstein equations for diagonal metrics are integrable*, similarly to the $1+1$ vacuum Einstein equations [8,9].

2. The field equations

For the diagonal metric (3), the inverse metric is

$$g^{\mu\nu} = \frac{1}{(H_\mu)^2}\delta^{\mu\nu} \tag{5}$$

and the Christoffel symbols are

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= 0 \\ \Gamma^\mu_{\mu\nu} &= \partial_\nu\left(\ln H_\mu\right) \\ \Gamma^\nu_{\mu\mu} &= -\frac{1}{(H_\nu)^2}H_\mu\partial_\nu H_\mu \end{aligned} \tag{6}$$

where μ, ν, λ are assumed to be mutually exclusive indices ($\mu \neq \nu$, $\mu \neq \lambda$, $\nu \neq \lambda$). Define the rotation coefficients

$$Q_{\mu\nu} = \frac{1}{H_\nu}\partial_\nu H_\mu \tag{7}$$

with which one can write the off-diagonal Ricci curvature tensor as

$$R_{\mu\nu} = -\sum_{\lambda\neq\mu,\nu}\frac{H_\mu}{H_\lambda}\left(\partial_\nu Q_{\lambda\mu} - Q_{\lambda\nu}Q_{\nu\mu}\right) \tag{8}$$

for $\mu \neq \nu$. As for the diagonal elements, the Ricci tensor gives

$$R_{\mu\mu} = -\sum_{\nu\neq\mu}\frac{H_\mu}{H_\nu}E_{\mu\nu} \tag{9}$$

where

$$E_{\mu\nu} = \partial_\nu Q_{\mu\nu} + \partial_\mu Q_{\nu\mu} + \sum_{\lambda\neq\mu,\nu} Q_{\mu\lambda}Q_{\nu\lambda}. \tag{10}$$

The scalar curvature is

$$R = -2\sum_{\mu<\nu}\frac{E_{\mu\nu}}{H_\mu H_\nu}. \tag{11}$$