Journal of Geometry and Physics 80 (2014) 49–57

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

The "ghost" symmetry in the CKP hierarchy

Jipeng Cheng [a], Jingsong He [b,*]

[a] Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, PR China

[b] Department of Mathematics, Ningbo University, Ningbo, Zhejiang 315211, PR China

ARTICLE INFO

Article history:
Received 22 May 2013
Received in revised form 7 November 2013
Accepted 26 February 2014
Available online 5 March 2014

MSC:
35Q53
37K10
37K40

Keywords:
CKP hierarchy
Squared eigenfunction symmetry
Spectral representation
Squared eigenfunction potential

ABSTRACT

In this paper, we systematically study the "ghost" symmetry in the CKP hierarchy through its actions on the Lax operator, dressing operator, eigenfunctions and the tau function. In this process, the spectral representation of the eigenfunction is developed and the squared eigenfunction potential is investigated.

1. Introduction

In this paper, given any pseudo-differential operator $A = \sum_i a_i \partial^i$ with $\partial = \partial_x$ and any function f,

$$A_+ = \sum_{i\geq 0} a_i\partial^i, \qquad A_- = \sum_{i<0} a_i\partial^i, \qquad \mathrm{Res}\left(\sum_i a_i\partial^i\right) = a_{-1}, \qquad \left(\sum_i a_i\partial^i\right)_{[k]} = a_k, \qquad A^* = \sum_i (-\partial)^i a_i, \tag{1}$$

and $A(f)$ denotes the action of A on f.

The Kadomtsev–Petviashvili (KP) hierarchy [1] is an important research object in the area of mathematical physics, which is defined by the following Lax equation

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, 3, \ldots, \tag{2}$$

with the Lax operator L given by

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots, \tag{3}$$

where the coefficient functions u_i are all the functions of the time variables $t = (t_1 = x, t_2, t_3, \ldots)$. The Lax operator (3) can be generated by the dressing operator $\Phi = 1 + \sum_{k=1}^{\infty} a_k\partial^{-k}$ in the following way:

$$L = \Phi\partial\Phi^{-1}. \tag{4}$$

* Corresponding author.
E-mail address: hejingsong@nbu.edu.cn (J. He).

http://dx.doi.org/10.1016/j.geomphys.2014.02.008
0393-0440/

Then the Lax equation (11) can also be expressed as Sato's equation

$$\frac{\partial \Phi}{\partial t_n} = -(L^n)_- \Phi, \quad n = 1, 2, 3, \ldots. \tag{5}$$

Another important object is the Baker–Akhiezer (BA) wave function $\psi_{BA}(t, \lambda)$ defined via:

$$\psi_{BA}(t, \lambda) = \Phi(e^{\xi(t,\lambda)}) = \phi(t, \lambda)e^{\xi(t,\lambda)} \tag{6}$$

with $\xi(t, \lambda) \equiv \sum_{k=1}^{\infty} t_k\lambda^k$ and $\phi(t, \lambda) = 1 + \sum_{i=1}^{\infty} a_i(t)\lambda^{-i}$, which satisfies

$$L(\psi_{BA}(t, \lambda)) = \lambda\psi_{BA}(t, \lambda), \qquad \partial_{t_n}\psi_{BA}(t, \lambda) = (L^n)_+(\psi_{BA}(t, \lambda)), \quad n = 1, 2, 3, \ldots. \tag{7}$$

The adjoint BA function $\psi^*_{BA}(t, \lambda)$ is introduced through the following way

$$\psi^*_{BA}(t, \lambda) = \Phi^{*-1}(e^{-\xi(t,\lambda)}). \tag{8}$$

The KP hierarchy can also be expressed in terms of a single function called the tau function $\tau(t)$ [1], which is related with the wave function in the way below,

$$\psi_{BA}(t, \lambda) = \frac{\tau(t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, t_3 - \frac{1}{3\lambda^3}, \ldots)}{\tau(t_1, t_2, t_3, \ldots)} e^{\xi(t,\lambda)}. \tag{9}$$

Because of the existence of the tau function, many important results in the KP hierarchy can be considered in terms of the tau function, such as the flow equation, Hirota's bilinear equation and algebraic constraint. KP hierarchy has two famous sub-hierarchies [1,2]: the BKP hierarchy and the CKP hierarchy. Just like the KP hierarchy, the BKP hierarchy also owns one single tau function, which bring much convenience to the study of the BKP hierarchy.

The CKP hierarchy [2] is a reduction of the KP hierarchy through the constraint on L given by (3) as

$$L^* = -L, \tag{10}$$

then L is called the Lax operator of the CKP hierarchy, and the associated Lax equation of the CKP hierarchy is

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 3, 5, \ldots, \tag{11}$$

which compresses all even flows, i.e., the Lax equation of the CKP hierarchy has only odd flows. The CKP constraint (10) on the corresponding dressing operator Φ will be $\Phi^* = \Phi^{-1}$. And thus in the CKP hierarchy $\psi^*_{BA}(t, \lambda) = \Phi^{*-1}(e^{-\xi(t,\lambda)}) = \Phi(e^{-\xi(t,\lambda)}) = \psi_{BA}(t, -\lambda)$. So in the CKP hierarchy, it is enough to only study the wave function $\psi_{BA}(t, \lambda)$. The CKP hierarchy (11) is equivalent to the following bilinear equation:

$$\int d\lambda\, \psi_{BA}(t, \lambda)\psi_{BA}(t', -\lambda) = 0, \tag{12}$$

where $\int d\lambda \equiv \oint_\infty \frac{d\lambda}{2\pi i} = \mathrm{Res}_{\lambda=\infty}$. By now, the CKP hierarchy has attracted many researches [3–13].

In contrast to the KP and the BKP cases, there seems not a single tau function to describe the CKP hierarchy in the form of Hirota bilinear equations (that is, Hirota's equations are no longer of the type $P(D)\tau \cdot \tau = 0$) [2]. The existence of this kind of tau function for the CKP hierarchy is a long-standing problem. Much work has been done in this field [3–6]. Since the existence of tau function of the CKP hierarchy is not proved, many important results on the Lax operator and dressing operator of the CKP hierarchy cannot be transferred to the tau function, until Chang and Wu [6] introduces a kind of tau function $\tau_c(t)$ for the CKP case, which is related with the wave function in the following way

$$\begin{aligned}\psi_{BA}(t, \lambda) &= \sqrt{\varphi(t, \lambda)}\frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)} e^{\xi(t,\lambda)} \\ &= \left(1 + \frac{1}{\lambda}\partial_x \log \frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)}\right)^{1/2} \frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)} e^{\xi(t,\lambda)},\end{aligned} \tag{13}$$

where $[\lambda^{-1}] = (\lambda^{-1}, \frac{1}{3}\lambda^{-3}, \ldots)$ and $\varphi(t, \lambda) = \phi(t, \lambda)\phi(t - 2[\lambda^{-1}], -\lambda)$. Note that the relation between the tau function and the wave function is different from the cases of KP and BKP, because there is a square-root factor depending on the tau function. And the relation of the new CKP tau function τ_c to the (C-reduced) KP tau function τ is shown in the Appendix A.

The "ghost" symmetry [14–18], sometimes called the squared eigenfunction symmetry [14], is one of the most important symmetries in the integrable system, which is defined through the squared eigenfunctions [14]. The usage of the squared eigenfunction to construct the symmetry flows can be traced back to [19–21], where operators $\psi\partial^{-1}\psi^*$ were introduced to construct $L - A$ pairs for symmetry flows (ψ and ψ^* being wave functions respectively of L and L^* operators). And similar symmetry flows are also studied from the Hamiltonian point of view [22]. The "ghost" symmetry can be used to define the new integrable system, such as the symmetry constraint [15,17,23–28] and the extended integrable systems [29,30],