Physica D 320 (2016) 9–18

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Symmetry types and phase-shift synchrony in networks

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HIGHLIGHTS

- Comparisons between H/K lists for equations, oscillators, and systems are made.
- Comparisons of phase-shift patterns are made.
- The H/K theorem for periodic solutions of coupled oscillators is valid.
- The H/K lists for equivariant and admissible maps are not equal for coupled equations.

ARTICLE INFO

Article history:
Received 6 May 2015
Received in revised form
12 December 2015
Accepted 14 December 2015
Available online 31 December 2015
Communicated by K. Josic

Keywords:
Phase-shift synchrony
Periodic solutions
Coupled cell networks
Symmetry

ABSTRACT

In this paper we discuss what is known about the classification of symmetry groups and patterns of phase-shift synchrony for periodic solutions of coupled cell networks. Specifically, we compare the lists of spatial and spatiotemporal symmetries of periodic solutions of admissible vector fields to those of equivariant vector fields in the three cases of $\mathbb{R}^n$ (coupled equations), $\mathbb{T}^n$ (coupled oscillators), and $(\mathbb{R}^k)^n$ where $k \geq 2$ (coupled systems). To do this we use the H/K Theorem of Buono and Golubitsky (2001) applied to coupled equations and coupled systems and prove the H/K theorem in the case of coupled oscillators. Josić and Török (2006) prove that the H/K lists for equivariant vector fields and admissible vector fields are the same for transitive coupled systems. We show that the corresponding theorem is false for coupled equations. We also prove that the pairs of subgroups $H \supset K$ for coupled equations are contained in the pairs for coupled oscillators which are contained in the pairs for coupled systems. Finally, we prove that patterns of rigid phase-shift synchrony for coupled equations are contained in those of coupled oscillators and those of coupled systems.

1. Introduction

Many biological phenomena, such as respiration [1,2], locomotion [3–7], or rivalry [8] are characterized by robust rhythmic patterns that exhibit particular phase relationships or phase-shifts. The neuronal networks responsible for these behaviors can be represented as coupled systems of differential equations that exhibit periodic behavior corresponding to these phase-shift patterns. These phase relationships appear to occur robustly in nature; hence, it is reasonable to utilize models in which the phase-shifts are rigid, that is, they are preserved under small perturbations of the corresponding network of differential equations.

It is well known that rigid phase-shifts in networks of differential equations may be caused by the symmetries of the underlying network. Stewart et al. [9–11] developed a framework for studying coupled systems of differential equations that associates to each directed graph a collection of admissible vector fields. These authors and others have then studied properties of solutions and of bifurcations in admissible systems.

The fact that rigid phase-shifts are also informed by the state spaces of the network nodes is frequently overlooked. Classically, modelers often use state spaces for individual nodes that are either one-dimensional $\mathbb{R}$ (smoothed out integrate and fire systems), circles $\mathbb{T}$ (oscillators), or multidimensional $\mathbb{R}^k$ where $k \geq 2$ (Hodgkin–Huxley neurons). We call these cases: *coupled equations*, *coupled oscillators*, and *coupled systems*. This paper studies the similarities and the differences between patterns of phase-shift synchrony forced by network symmetry in these three contexts, both for the class of admissible vector fields and for the less restrictive class of equivariant vector fields.

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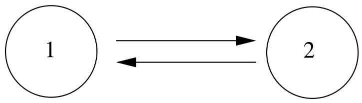
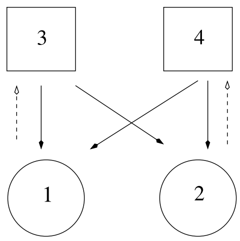
http://dx.doi.org/10.1016/j.physd.2015.12.005
0167-2789/



(a) Two identical node, identical coupling network with $\sigma = (1\ 2)$ symmetry.



(b) Four-cell network with $\sigma = (1\ 2)(3\ 4)$ symmetry.

Fig. 1. Examples of networks with $\mathbf{Z}_2$ symmetry.

Examples of rigid phase-shifts for periodic solutions

Before presenting our results we recall some terminology. Specifically, we discuss rigid phase-shifts, patterns of phase-shift synchrony, and their relationship to symmetry. Suppose that a network has n nodes and state variables $(x_1, \ldots, x_n)$.

Definition 1.1. A T-periodic solution $X(t) = (x_1(t), \ldots, x_n(t))$ of an admissible system has a *phase-shift* θ_{ij} if there are two nodes i and j such that

$$x_j(t) = x_i(t + \theta_{ij}T).$$

This phase-shift is *rigid* if any perturbed admissible system has a perturbed $\tilde{T}$-periodic solution $\tilde{X}(t) = (\tilde{x}_1(t), \ldots, \tilde{x}_n(t))$ such that

$$\tilde{x}_j(t) = \tilde{x}_i(t + \theta_{ij}\tilde{T})$$

with the same phase-shift.

To illustrate different rigid phase-shifts, consider the networks shown in Fig. 1. The admissible systems corresponding to Fig. 1(a) have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1) \end{aligned} \tag{1.1}$$

and the admissible systems corresponding to Fig. 1(b) have the form

$$\begin{aligned} \dot{x}_1 &= h(x_1, \overline{x_3, x_4}) \\ \dot{x}_2 &= h(x_2, \overline{x_3, x_4}) \\ \dot{x}_3 &= g(x_3, x_1) \\ \dot{x}_4 &= g(x_4, x_2) \end{aligned} \tag{1.2}$$

where $x_1, x_2 \in P$; $x_3, x_4 \in Q$ and P and Q are phase spaces of individual nodes. The overline indicates that $h(a, b, c) = h(a, c, b)$.

As is well known, there are stable anti-phase periodic solutions of coupled systems and coupled oscillators for (1.1) having the form

$$x_2(t) = x_1\left(t + \frac{1}{2}T\right),$$

(found by Hopf bifurcation), but it is less often discussed that such solutions cannot exist for coupled equations. Similarly, there are stable periodic solutions of coupled systems and of coupled oscillators for (1.2) having the form

$$x_2(t) = x_1\left(t + \frac{1}{2}T\right) \qquad x_4(t) = x_3\left(t + \frac{1}{2}T\right),$$

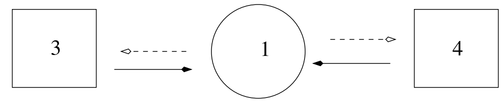


Fig. 2. $\mathbf{Z}_2$-symmetric, three-cell quotient network of the network in Fig. 1(b).

but this solution does not exist for coupled equations. Each of these solution types is generated by the σ symmetry of their associated networks (see Fig. 1).

The second network illustrates the subtle fact that a pattern of phase-shift synchrony can be forced by a symmetry on a quotient network, rather than by a symmetry on the network itself. See [12–14]. Specifically, (1.2) can also have a periodic solution of the form

$$x_2(t) = x_1(t) \qquad x_4(t) = x_3\left(t + \frac{1}{2}T\right) \tag{1.3}$$

that is not generated by a network symmetry of Fig. 1(b). Note that $\Delta = \{x_1 = x_2\}$ is a flow-invariant subspace for every admissible vector field in (1.2) and the equations for the admissible vector fields restricted to Δ have the form

$$\begin{aligned} \dot{x}_1 &= h(x_1, \overline{x_3, x_4}) \\ \dot{x}_3 &= g(x_3, x_1) \\ \dot{x}_4 &= g(x_4, x_1). \end{aligned} \tag{1.4}$$

These equations correspond to the *quotient* network given in Fig. 2 and this quotient network has a symmetry $\tau = (3\ 4)$. It is the symmetry τ on the quotient network that generates the solution type (1.3). Specifically, stable solutions where $x_3(t)$ and $x_4(t)$ are in anti-phase can be found by Hopf bifurcation for coupled systems and numerically for coupled oscillators. In these solutions $x_1(t)$ oscillates at twice the frequency of $x_3(t)$. It follows that in network 1(b), $x_2(t) = x_1(t + \frac{1}{2}T) = x_1(t)$.

Patterns of phase-shift synchrony

In a series of papers, Stewart and Parker [15,16] and Golubitsky, Romano, and Wang [17,18] proved that in path connected networks of either coupled equations or coupled systems, rigid phase-shifts always result from symmetry. However, that symmetry may be a symmetry of a quotient network, rather than a network symmetry.

More precisely, we define:

Definition 1.2. A *pattern of phase-shift synchrony* is a subset of pairs of nodes i and j and phase-shifts $0 \leq \theta_{ij} < 1$. A T-periodic solution $x(t) = (x_1(t), \ldots, x_n(t))$ *exhibits* this pattern of synchrony if

$$x_j(t) = x_i(t + \theta_{ij}T)$$

for all designated pairs i, j and the θ_{ij} are rigid.

These four papers [15–18] prove the following: Suppose that a periodic solution $X(t)$ exhibits a pattern of phase-shift synchrony. Then the polydiagonal defined by

$$\triangle = \{X = (x_1, \ldots, x_n) : x_i = x_j \text{ when } \theta_{ij} = 0\}$$

is flow-invariant. Moreover, there is a cyclic symmetry τ on the quotient network corresponding to $\triangle$ that generates all of the nonzero θ_{ij} in the pattern of phase-shift synchrony.

Symmetry groups of periodic solutions for equivariant systems

A *symmetry* of a system of differential equations

$$\dot{X} = F(X) \tag{1.5}$$