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Turbulence properties and global regularity of a modified Navier–Stokes equation

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HIGHLIGHTS

- We introduce a new nonlocal equation similar to the Navier–Stokes equation.
- The inviscid version of this new equation possesses an infinite number of conserved quantities.
- We prove global regularity for this new equation.
- Turbulence properties lie between those of the Burgers and Naver–Stokes equations.

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ABSTRACT

We introduce a modification of the Navier–Stokes equation that has the remarkable property of possessing an infinite number of conserved quantities in the inviscid limit. This new equation is studied numerically and turbulence properties are analyzed concerning energy spectra and scaling of structure functions. The dissipative structures arising in this new equation are curled vortex sheets instead of the vortex tubes arising in Navier–Stokes turbulence. The numerically calculated scaling of structure functions is compared with a phenomenological model based on the She–Lévêque approach.

Finally, for this equation we demonstrate global well-posedness for sufficiently smooth initial conditions in the periodic case and in $\mathbb{R}^3$. The key feature is the availability of an additional estimate which shows that the L^4-norm of the velocity field remains finite.

1. Introduction

In this paper, we introduce a new equation which is a hybrid of the Navier–Stokes equation and the Burgers equation. Our goal is to show the existence of multi-dimensional model equations which possess a direct turbulent cascade to small scales, non-trivial intermittency, nonlocal interaction and yet an infinite number of conserved quantities. To our knowledge, there seems to be no model equation in the literature with these desirable properties. Our model could therefore be seen as an interesting starting point for testing numerous methods like e.g. phenomenological approaches or methods from field theory (Martin–Siggia–Rose formalism [1], instantons [2], OPE [3]), in a multi-dimensional setup. Our new equation may play the same role as the 1D-Burgers equation in higher dimensions.

Turbulence properties of this equation are analyzed using numerical simulations. We calculate energy spectra and scaling of higher order structure functions. A key observation from the numerical simulations of this modified Navier–Stokes equation is that the most dissipative structures consist of curled vortex sheets instead of the vortex tubes in conventional Navier–Stokes turbulence. Using this information, a She–Lévêque type model [4] is derived and compared to the numerically obtained scaling of higher order structure functions.

In addition, for this new equation we can show existence and regularity for H^1 initial conditions of arbitrary size. This will be carried out in $\mathbb{R}^3$ and periodic domains in $\mathbb{R}^3$. The simple modification of the nonlinearity makes the proof of global solutions possible, insofar as an additional estimate is available showing that the solution remains finite in L^p, $2 < p < \infty$. With $p = 4$, this is then coupled with standard estimates for the H^1-norm to complete the proof. In contrast to other approaches for regularization of the Navier–Stokes equation using dispersive mollification [5–7] acting on small scales, our modification acts on all spatial scales.

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The outline of this paper is as follows: in Section 2 we motivate and introduce our new equation. Turbulence statistics and phenomenological modeling are considered in Section 3. Section 4 contains the proof of existence of global solutions. We finish with remarks on possible further consequences of the existence of an infinite number of conserved quantities.

2. Model equation

We consider a three-dimensional domain Ω which shall be either $\mathbb{R}^3$ or a bounded cube in $\mathbb{R}^3$ with periodic boundary conditions. Let $\mathrm{P} = 1 - \Delta^{-1}\nabla \otimes \nabla$ be the Leray–Hopf projection operator (with periodic boundary conditions when Ω is bounded):

$$\mathrm{P}[\mathrm{P}[\mathbf{u}]] = \mathrm{P}[\mathbf{u}], \qquad \nabla \cdot \mathrm{P}[\mathbf{u}] = 0. \tag{1}$$

The usual incompressible Navier–Stokes equation

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{f}, \qquad \nabla\cdot\mathbf{v} = 0 \tag{2}$$

can be written with the projection operator P

$$\frac{\partial}{\partial t}\mathbf{v} + \mathrm{P}[\mathbf{v}\cdot\nabla\mathbf{v}] = \nu\Delta\mathbf{v} + \mathrm{P}[\mathbf{f}], \qquad \nabla\cdot\mathbf{v} = 0$$

so that no explicit pressure term is present in the equation.

We can rewrite the Navier–Stokes equation without the incompressibility constraint in the form

$$\frac{\partial}{\partial t}\mathbf{u} + \mathrm{P}[\mathbf{u}]\cdot\nabla\mathrm{P}[\mathbf{u}] = \nu\Delta\mathbf{u} + \mathbf{f}, \tag{3}$$

where the solution of the Navier–Stokes equation can be recovered by taking $\mathbf{v} = \mathrm{P}[\mathbf{u}]$.

Eq. (3) can be compared with the Burgers equation whose structure is formally similar:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} + \mathbf{f}. \tag{4}$$

For Eq. (4) the nonlinearity is purely local, whereas for Eq. (3) the nonlinear interaction involves the nonlocal projection.

A natural hybrid of these two equations leads a new model equation involving a compressible velocity field $\mathbf{u}$ that is convected by its solenoidal part $\mathrm{P}[\mathbf{u}]$:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathrm{P}[\mathbf{u}]\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} + \mathbf{f}. \tag{5}$$

More accurately this means that the convection of the velocity field $\mathbf{u}$ is local in position space, but the projection operator is local in Fourier space and thus shares this mixture of local and nonlocal interactions with the original Navier–Stokes equation.

Writing this equation in the more conventional form of a solenoidal velocity field $\mathbf{v}$,

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla\left(\frac{\partial}{\partial t}\varphi - \nu\Delta\varphi\right) + \mathbf{v}\cdot\nabla\nabla\varphi = \nu\Delta\mathbf{v}, \tag{6}$$

where the compressible vector $\mathbf{u}$ is decomposed as $\mathbf{u} = \mathbf{v} + \nabla\varphi$, the similarity as well as the difference to the original Navier–Stokes equation (2) is stressed: the gradient term $\nabla\left(\frac{\partial}{\partial t}\varphi - \nu\Delta\varphi\right)$ corresponds to the pressure contribution ∇p whereas the additional term $\mathbf{v}\cdot\nabla\nabla\varphi$ forms the difference to the Navier–Stokes equation.

3. Turbulence statistics

By construction the presented model equation is an intermediate step between the Navier–Stokes and Burgers equation, which in turn differ significantly in their dynamical evolution and turbulent behavior. In Navier–Stokes turbulence, on the one hand, the most dissipative structures are vortex filaments, while for Burgers equation shocks dominate the turbulent flow. It is of obvious interest in how far our model equation bridges between those, which structures are the most dominant for turbulent flows and how these structures influence the turbulence statistics. We therefore extend the She–Lévêque reasoning, which describes Navier–Stokes and Burgers turbulence well, to our model equation and test it against numerical simulations by comparing the scaling exponents of the structure functions.

Numerical simulations are carried out with a second-order in space finite difference scheme with a strongly stable third-order Runge–Kutta time integration with resolutions up to 512^3. The initial conditions were chosen as Orszag–Tang-like (see [8]) large-scale perturbations:

$$\begin{aligned}
u_x &= A\left(-2\sin(2y) + \sin(z) + 2\cos(2y) + \cos(z)\right)\\
u_y &= A\left(-2\sin(x) + \sin(z) + 2\cos(x) + \cos(z)\right)\\
u_z &= A\left(\sin(x) + \sin(y) - 2\cos(2x) + \cos(y)\right).
\end{aligned}$$

For simplicity and comparability both velocity and its solenoidal projection are set to equal values. The physical domain stretches from $-\pi$ to π; the above defined conditions, thus, are both large-scale perturbations and periodic. All hydrodynamical models will be simulated in comparison, using these initial conditions. We consider only decaying turbulence without external forces. For the parameters of all performed runs see Table 1, which shows the numerical value of the quantities at the time of maximum enstrophy $t = t_{\mathcal{E}}$.

Fig. 1 shows the decay of kinetic energy for the considered hydrodynamical models in comparison. The tendency of Burgers turbulence to form shocks and the dissipative nature of these structures lead to a faster energy decay compared to the Navier–Stokes equation. The new model equation exhibits a less violent form of dissipation; its energy decay lies in between the others. The difference in turbulence development is identified in a more precise way when comparing the time $t_{\mathcal{E}}$ of maximum enstrophy $\mathcal{E} = \int_\Omega \boldsymbol{\omega}^2 \mathrm{d}x$. As Fig. 1 (right) indicates, the enstrophy of Burgers turbulence reaches its peak significantly faster than for the Navier–Stokes equation, in which vortex filaments dominate the turbulent flow. The proposed model equation ranges between them. This hints at the development of coherent structures at a timescale slower than shock-formation of Burgers equation but faster than the formation of vortex tubes for the Navier–Stokes equation. Precise values for the timescales are stated in Table 1.

Fig. 2 depicts a volume render of the fully developed turbulence, the snapshot in each case taken at $t = t_{\mathcal{E}}$. As expected, the Burgers flow (left) is dominated by shocks and the Navier–Stokes flow (right) consists of vortex filaments. For the proposed model equation (middle), the most dominant structures are two-dimensional folded vortex-sheets.

A phenomenological description, which takes into account the most dissipative structures of the turbulent flow, is the model of She and Lévêque [4] connected to log-Poisson statistics of the local energy dissipation [9]. They state that the scaling exponent ζ_p of the p-th structure function behaves like

$$\zeta_p = \frac{(1-k)p}{3} + C_0\left(1 - \left(\frac{C_0 - k}{C_0}\right)^{\frac{p}{3}}\right), \tag{7}$$

where C_0 is the co-dimension of the most dissipative structures in the evolved flow and k is the time-scaling exponent. This formula will be referred to as the *She–Lévêque model* even though in [4] it is applied exclusively to the Navier–Stokes equation (where $C_0 = 2,\ k = 2/3$).