Journal of Geometry and Physics 79 (2014) 14–28

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Free boundary stable hypersurfaces in manifolds with density and rigidity results

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ARTICLE INFO

Article history:
Received 11 November 2013
Accepted 29 January 2014
Available online 6 February 2014

MSC:
53A10
53C24

Keywords:
Manifolds with density
Free boundary
Mean curvature
Stability
Rigidity

ABSTRACT

Let M be a weighted manifold with boundary ∂M, i.e., a Riemannian manifold where a density function is used to weight the Riemannian Hausdorff measures. In this paper we compute the first and second variational formulas of the interior weighted area for deformations by hypersurfaces with boundary in ∂M. As a consequence, we obtain variational characterizations of critical points and second order minima of the weighted area with or without a volume constraint. Moreover, in the compact case, we obtain topological estimates and rigidity properties for free boundary stable and area-minimizing hypersurfaces under certain curvature and boundary assumptions on M. Our results and proofs extend previous ones for Riemannian manifolds (constant densities) and for hypersurfaces with empty boundary in weighted manifolds.

1. Introduction

Stable hypersurfaces in a Riemannian manifold with boundary are second order minima of the *interior area* for compactly supported deformations preserving the boundary of the manifold and, possibly, the volume separated by the hypersurface. From the first variation formulas [1] such hypersurfaces have constant mean curvature and free boundary meeting orthogonally the boundary of the manifold. Moreover, the second variation formula [1] implies that the associated index form is nonnegative for functions with compact support (and mean zero if the volume-preserving condition is assumed). The stability property has been extensively discussed and plays a central role in relation to classical minimization problems such as the Plateau problem or the isoperimetric problem.

The study of variational questions associated to the area functional in *manifolds with density*, also called *weighted manifolds* or *smooth mm-spaces*, has been a focus of attention in the last years. A *manifold with density* is a connected Riemannian manifold, possibly with boundary, where a smooth positive function is used to weight the Hausdorff measures associated to the Riemannian distance. This kind of structures has been considered by many authors and provides a generalization of Riemannian geometry which is currently of increasing interest. For a nice introduction to weighted manifolds we refer the reader to Chapter 18 of Morgan's book [2] and to Chapter 3 of Bayle's thesis [3]. In the present paper we study *free boundary stable hypersurfaces* in manifolds with density, by obtaining variational characterizations, topological and geometrical information, and rigidity results for the ambient manifold. In order to describe our results in more detail we need to introduce some notation and definitions.

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http://dx.doi.org/10.1016/j.geomphys.2014.01.013

Let M be a Riemannian manifold endowed with a density $f = e^{\psi}$. We consider a smooth oriented hypersurface Σ immersed in M in such a way that $\mathrm{int}(\Sigma) \subset \mathrm{int}(M)$ and $\partial\Sigma \subset \partial M$ whenever $\partial\Sigma \neq \emptyset$. We say that Σ is *strongly f-stationary* if it is a critical point of the *weighted area functional* under compactly supported deformations moving ∂M along ∂M. Note that the *weighted area* $A_f(\Sigma)$ defined in (2.1) is relative to the interior of M, so that $\Sigma \cap \partial M$ does not contribute to $A_f(\Sigma)$. If, in addition, the hypersurface Σ has non-negative second derivative of the weighted area for any variation, then we say that Σ is *strongly f-stable*. In the Riemannian setting (constant density $f = 1$), these definitions coincide with the classical notions of free boundary minimal and stable hypersurfaces. Recently, many authors have considered complete strongly f-stable hypersurfaces *with empty boundary*, see [4–8] and [9], among others. However, not much is known about strongly f-stable hypersurfaces *with non-empty boundary*, and this has been in fact our main motivation in the present work.

Our first aim in this paper is to provide variational characterizations of strongly f-stationary and stable hypersurfaces in the same spirit of the ones given by Ros and Vergasta in [1] for the Riemannian case. This is done in Section 3, where we follow the arguments for hypersurfaces with empty boundary in [3, Ch. 3] and [10], in order to compute the first and second derivatives of the weighted area. As a consequence, we deduce that a hypersurface Σ in M is strongly f-stationary if and only if it has vanishing f-mean curvature and meets ∂M orthogonally in the points of $\partial\Sigma$, see Corollary 3.3. The *f-mean curvature* of Σ is the function H_f in (2.4) previously introduced by Gromov [11] in relation to the first derivative of the weighted area, see Lemma 3.2. We also show that the strong f-stability of Σ is equivalent to that the associated f-index form defined in (3.3) is nonnegative for smooth functions with compact support, see Corollary 3.6. At this point, it is worth mentioning that the techniques employed in this section allow also to characterize critical points and second order minima of the weighted area for deformations *preserving the weighted volume* V_f defined in (2.1). This is closely related to the *partitioning problem*, which consists of separating a given weighted volume in M with the least possible interior weighted area. However, besides showing some relevant situations in Examples 3.4, 3.7 and 3.8, the partitioning problem and the associated f-stable hypersurfaces will not be treated in detail. Some characterization results for compact f-stable hypersurfaces with free boundary in a Euclidean solid cone where a homogeneous density is considered can be found in [12].

In the remainder of the paper we mainly investigate the relationship between the topology of compact strongly f-stable hypersurfaces and the geometry of the ambient manifold by means of the second variation formula. As a motivation, note that the f-index form in (3.3) of a hypersurface Σ is a quadratic form which involves the extrinsic geometry of Σ, the second fundamental form II of ∂M, and the *Bakry–Émery–Ricci curvature* Ric_f of M defined in (2.2). The 2-tensor Ric_f was first introduced by Lichnerowicz [13,14], and later generalized by Bakry and Émery [15] in the framework of diffusion generators. In particular, it is easy to observe that the stability inequality becomes more restrictive provided II and Ric_f are always semidefinite positive. Hence *local convexity* of ∂M and *nonnegativity of the Bakry–Émery–Ricci curvature* become natural hypotheses in order to obtain interesting consequences from the stability condition.

In Section 4.1 we establish some results in this direction. In fact, by assuming $\mathrm{Ric}_f \geqslant 0$ and $\mathrm{II} \geqslant 0$ we deduce in a quite straightforward way that a compact strongly f-stable hypersurface must be totally geodesic, see Lemma 4.1. Moreover, we also have $\mathrm{Ric}_f(N, N) = 0$ and $\mathrm{II}(N, N) = 0$, where N is the unit normal to Σ. In particular, if $\mathrm{Ric}_f > 0$ or $\mathrm{II} > 0$, then there are no compact strongly f-stable hypersurfaces in M. This property was observed by Simons [16] for the Riemannian case, and later generalized by Fan [4], and Cheng, Mejia and Zhou [7] for hypersurfaces with empty boundary in manifolds with density. On the other hand, Espinar showed in [8] that Lemma 4.1 also holds for complete strongly f-stable hypersurfaces of finite type and empty boundary.

The simplest examples of strongly f-stable totally geodesic hypersurfaces satisfying $\mathrm{Ric}_f(N, N) = 0$ and $\mathrm{II}(N, N) = 0$ are the horizontal slices $\{s\} \times \Sigma$ in a Riemannian product $\mathbb{R} \times \Sigma$, where Σ is a compact Riemannian manifold of non-negative Ricci curvature, and the logarithm of the density f is a linear function in $\mathbb{R}$. These are not the unique examples we may give, i.e., the existence of compact strongly f-stable hypersurfaces in the above conditions does not imply that the metric of M splits, even locally, as a product metric, see [17]. However, in Theorem 4.2 we prove the following rigidity result:

> *If a weighted manifold M with non-negative Bakry–Émery–Ricci curvature and locally convex boundary contains a compact, oriented, embedded, locally weighted area-minimizing hypersurface Σ with non-empty boundary, then there is a neighborhood of Σ in M isometric to a Riemannian product $(-\varepsilon_0, \varepsilon_0) \times \Sigma$.*

This local result can be globalized by means of a standard continuation argument. As a consequence, if we further assume that M is complete and *Σ minimizes the weighted area in its isotopy class*, then M is a Riemannian quotient of $\mathbb{R} \times \Sigma$. We must remark that this rigidity result was previously obtained by Liu [9] for weighted area-minimizing hypersurfaces *with empty boundary*. To prove it, Liu used the second variation formula to analyze the weighted area functional for the deformation by normal geodesics leaving from Σ. In our context, however, a normal geodesic starting from $\partial\Sigma$ is not necessarily confined to stay in M, and so this deformation cannot be considered. As we will explain in more detail later, this difficulty is solved by taking another deformation which moves $\partial\Sigma$ along ∂M.

In Section 4.2 we provide a topological restriction for strongly f-stable surfaces, and a rigidity result for weighted area-minimizing surfaces in a weighted 3-manifold M of *non-negative Perelman scalar curvature* and *f-mean convex boundary*. On the one hand, the *Perelman scalar curvature* S_f defined in (2.3) is the generalization of the Riemannian scalar curvature introduced by Perelman [18] when showing that the Ricci flow is a gradient flow. Let us indicate, as a remarkable difference with respect to the Riemannian case, that the Perelman scalar curvature *is not the trace* of the Bakry–Émery–Ricci curvature. In fact, we have the Bianchi identity $S_f = 2\,\nabla^*\mathrm{Ric}_f$, where ∇^* is the adjoint operator of ∇ with respect to the L^2-norm for