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# Codimension reduction in symmetric spaces

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ABSTRACT

In this paper we give a short geometric proof of a generalization of a well-known result about reduction of codimension for submanifolds of Riemannian symmetric spaces.



## 1. Introduction

The goal of this paper is to give a short geometric proof of the following generalization of the reduction of codimension theorem for submanifolds of space forms [1, page 339]:

**Theorem 1.** *Let $M$ be a submanifold of a symmetric space $\mathbb{S}$ and let $\nu(M)$ be its normal bundle. Assume that there exists a $\nabla^{\perp}$-parallel subbundle $\mathrm{V} \subset \nu(M)$ containing the first normal space, i.e. $\mathrm{N}^1 \subset \mathrm{V}$ where $\mathrm{N}^1 = \alpha(TM \times TM)$. If $TM \oplus \mathrm{V}$ is invariant by the curvature tensor of $\mathbb{S}$ then there exists a totally geodesic submanifold of $\mathbb{S}$ of dimension equal to* $\mathrm{rank}(TM \oplus \mathrm{V})$ *containing $M$.*

As a corollary of this result one can obtain several well-known special cases [2–8].

The hypothesis about the curvature invariance of $TM \oplus \mathrm{V}$ is redundant if $\mathbb{S}$ is a space form. We will give an example showing that such condition cannot be omitted in general, see Section 4.

Our proof of the above theorem was mainly inspired by the proof, due to C. Olmos, of the existence theorem of a totally geodesic submanifold with prescribed tangent space usually attributed to E. Cartan, see [9, Theorem 8.3.1, page 231]. Olmos' proof is based in Lemma 8.3.2 in [9, page 232]. We will need to use a slightly different version of this lemma which involves parallel translation along piece-wise smooth curves instead of smooth curves. We include it in an Appendix, with a sketch of its proof, for the sake of completeness.

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Theorem 1 does not hold for submanifolds of locally symmetric spaces. The problem is that under the same hypothesis, the "totally geodesic submanifold" containing $M$ may intersect itself, see example in Section 4. One can prove a slightly different version of this theorem for locally symmetric spaces by either assuming that the submanifold $M$ is embedded or allowing totally geodesic immersions (not necessarily 1–1) instead of totally geodesic submanifolds.

Finally, we want to point out that Theorem 1 can be obtained, besides our proof, by following two other different approaches. The first one makes use of the Grassmann bundle theory and the integration theory of differentiable distributions, see [10, Prop. 3, page 90]. The second one is based on a generalization of the classical theorem of existence and uniqueness of isometric immersions into space forms, see [11].

## 2. Basic definitions

We will say that a Riemannian manifold $M$ is a submanifold of a Riemannian manifold $\mathbb{S}$ if there is a 1–1 isometric immersion $f : M \to \mathbb{S}$. In order to simplify the notation, we shall assume that $M$ is a subset of $\mathbb{S}$, eventually endowed with a different topology, and $f$ is the inclusion map. If in addition $M$ has the induced topology from $\mathbb{S}$ we say that $M$ is an embedded submanifold.

We identify the tangent space to $M$ at a point $p$ with a subspace of $T_p\mathbb{S}$ and consider the orthogonal splitting $T_p\mathbb{S} = T_pM \oplus \nu_pM$. Here $\nu_pM$ is the normal space and $\nu(M)$ will denote the normal bundle of $M$.

We denote by $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{S}$ and by $\nabla$ and $\nabla^{\perp}$ the Levi-Civita and the normal connections of $M$ respectively. Let $\alpha$ and $A$ be the second fundamental form and shape operator of $M$ respectively. They are defined taking tangent and normal components by the Gauss and Codazzi formulas

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \qquad \overline{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi \tag{1}$$

and related by $\langle \alpha(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle$, for any tangent vector fields $X$ and $Y$ to $M$ and any normal vector field $\xi$.

## 3. Proof of Theorem 1

First notice that it suffices to prove the theorem locally around each point. Namely, to show the existence of a totally geodesic submanifold $N_p$ of $\mathbb{S}$ containing a neighborhood $U$ of $p$ in $M$ whose tangent space is $T_qN_p = T_qM \oplus \mathrm{V}_q$ for all $q \in U$. Indeed, the global result follows since a complete totally geodesic submanifold of a symmetric space $\mathbb{S}$ with a prescribed tangent space is unique as a global object [12, Lemma 2, page 235].

So we may assume that $M$ is small enough so that the normal exponential map $\exp^{\perp} : \mathrm{V}_0 \to \mathbb{S}$ is an immersion from a small neighborhood $\mathrm{V}_0$ of the zero section of V.

Set $N = \exp^{\perp}(\mathrm{V}_0)$. Since $M$ is the image of the zero section of V we get that $M$ is a submanifold of $N$. Now we are going to prove that $N$ is a totally geodesic submanifold by a similar argument as in the proof of Theorem 8.3.1 in [9, page 231]. It will suffice to prove that the parallel transport in $\mathbb{S}$ along any curve in $N$ preserves the tangent bundle $TN$. To do this we will fix a point $p \in M$ and we will show the following two properties:

(i) for any point $q$ in $N$, there is a curve $\gamma$ joining $p$ with $q$ such that the parallel transport along $\gamma$ in $\mathbb{S}$ of $T_pN$ is $T_{\gamma(t)}N$;
(ii) the tangent space $T_pN$ is preserved by parallel transport in $\mathbb{S}$ along any loop in $N$ based at $p$.

In order to prove (i), we will start showing that $TN$ is parallel with respect to the connection $\overline{\nabla}$ of $\mathbb{S}$ in directions tangent to $M$.

Since $TN_{|M} = TM \oplus \mathrm{V}$, a section $X$ of $TN_{|M}$ splits as $X = X_1 + X_2$, with $X_1 \in TM$ and $X_2 \in \mathrm{V}$. So if $v \in T_pM$, then

$$\overline{\nabla}_v X = \nabla_v X_1 + \alpha(v, X_1) + \nabla^{\perp}_v X_2 - A_{X_2} v.$$

This shows that $\overline{\nabla}_v X$ belongs to $TN$ since $\alpha(v, X_1) \in \mathrm{N}^1 \subset \mathrm{V}$ and V is parallel with respect to the normal connection of $M$.

The second step is to prove that $TN$ moves parallel along any normal geodesic $\gamma(t) = \exp_p(t\xi_p)$ for $p \in M$ and $\xi_p \in \mathrm{V}_0$. Observe that the tangent spaces to $N$ along $\gamma$ are generated by the Jacobi fields $J(t)$ along $\gamma(t)$ with initial conditions $J(0) \in T_{\gamma(0)}M$ and $J'(0) \in \mathrm{V}$.

Denote by $\mathrm{W}_t$ the parallel transport of $T_pN$ along $\gamma$ from $\gamma(0)$ to $\gamma(t)$.

Let $J(t)$ be any Jacobi vector field along $\gamma$ with $J(0) \in T_{\gamma(0)}M$ and $J'(0) \in \mathrm{V}_{\gamma(0)}$. Since $TM \oplus \mathrm{V}$ is invariant under the curvature tensor of the symmetric space $\mathbb{S}$ one gets that $J(t) \in \mathrm{W}_t$ for every $t$. Indeed, $\mathrm{W}_t$ is curvature invariant and so the Jacobi equation can be solved in $\mathrm{W}_t$. This shows that $T_{\gamma(t)}N \subset \mathrm{W}_t$, hence $\mathrm{W}_t = T_{\gamma(t)}N$, since both are linear spaces of the same dimension.

Now, if $q$ is any point in $N$, there exists a point $q_0$ in $M$ and a normal vector $\xi_{q_0} \in \mathrm{V}_0$ such that $q = \exp^{\perp}(\xi_{q_0})$. From the above discussion, any curve in $M$ connecting $p$ to $q_0$ followed by a normal geodesic from $q_0$ in the direction of $\xi_0$ gives one curve joining $p$ with $q$ satisfying (i).

Now we prove (ii) by using Lemma A.1 in Appendix (cf. [9, Lemma 8.3.2, page 232]). Let $c(s)$ be any loop in $N$ based at $p \in M$. There exists a loop $\hat{c}(s)$ in $M$ based at $p$ and a normal vector field $\xi(s) \in \mathrm{V}_{\hat{c}(s)}$ along $\hat{c}$ such that $c(s) = \exp^{\perp}(\xi(s))$.