Journal of Geometry and Physics 79 (2014) 34–52

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Equivariant, string and leading order characteristic classes associated to fibrations

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ARTICLE INFO

Article history:
Received 13 September 2013
Received in revised form 3 January 2014
Accepted 23 January 2014
Available online 6 February 2014

Keywords:
Characteristic classes
Infinite rank bundles
Equivariant index theorem
Gromov–Witten invariants
Loop group cohomology
Donaldson classes

ABSTRACT

Infinite rank vector bundles often appear as pushdowns of finite rank bundles from the total space of a fibration to the base space. The infinite rank bundles have string and leading order characteristic classes related to the characteristic classes of the finite rank bundles. We rewrite the S^1-index theorem as a statement about equivariant leading order classes on loop spaces, interpret certain Gromov–Witten invariants in terms of leading order and string classes, show that the generators of the cohomology of a loop group are Chern–Simons string classes, and relate Donaldson invariants to leading order currents.

1. Introduction

Infinite rank principal or vector bundles appear frequently in mathematical physics, even before quantization. For example, string theory involves the tangent bundle to the space of maps $\mathrm{Maps}(\Sigma^2, M)$ from a Riemann surface to a manifold M, while any gauge theory relies on the principal bundle $\mathcal{A}^* \to \mathcal{A}^*/\mathcal{G}$ of irreducible connections over the quotient by the gauge group. Finally, formal proofs of the Atiyah–Singer index theorem take place on the free loop space LM, and in particular use calculations on TLM. As explained below, many of these examples arise from pushing finite rank bundles on the total space of a fibration down to an infinite rank bundle on the base space.

For the correct choice of structure group, these infinite rank bundles can be topologically nontrivial. As for finite rank bundles, nontriviality is often detected by infinite dimensional analogs of the Chern–Weil construction of characteristic classes, as in [1–5], with a survey in [6]. The choice of structure group is determined by natural classes of connections on these bundles, which typically take values in either the Lie algebra of a gauge group or a Lie algebra of zeroth order pseudodifferential operators (ΨDOs). There are essentially three types of characteristic forms for these connections, one using the Wodzicki residue for ΨDOs, one using the zeroth order or leading order symbol, and one using integration over the fiber. The corresponding cohomology classes are called residue classes, leading order classes, and string classes, respectively. As shown in [7], the residue classes vanish, but nontrivial residue Chern–Simons classes exist [3].

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http://dx.doi.org/10.1016/j.geomphys.2014.01.011
0393-0440/

In this paper, we focus on gauge group connections and produce examples of nontrivial leading order and string characteristic classes for some infinite rank bundles associated to loop spaces, Gromov–Witten theory and gauge theory. While the residue classesare inherently infinite dimensional objects and difficult to compute, the leading order and string classes for infinite rank bundles on the base space of a fibration are often related to characteristic classes of the finite rank bundle on the total space. This makes the leading order and string classes more computable. In particular, in some cases we can relate the leading order classes to the string classes.

In Section 2, we describe the basic setup, which is well known from local proofs of the families index theorem. To a fibration $Z \to M \to B$ of closed manifolds and a bundle with connection $(E, \nabla) \to M$, one can associate an infinite rank bundle with connection $(\mathcal{E}, \nabla') \to B$. This is a gauge connection if the fibration admits a flat connection, for example if the fibration is trivial. In this case, we can define the associated leading order Chern classes of $\mathcal{E}$. Even if the fibration is not flat, $\mathcal{E}$ has string classes, which are topological pushdowns of the Chern classes of E. The leading order and string classes do not live in the same degrees. Both classes have associated Chern–Simons or transgression forms.

In Section 3, we show that the S^1 Atiyah–Singer index theorem can be rewritten as an equality involving leading order classes on the loop space LM of a closed manifold M (Theorem 3.4). (More precisely, we work with the version of the S^1-index theorem called the Kirillov formula in [8].) This is an attempt to mimic the formal proofs of the ordinary index theorem on loop space [9,10], but differs in significant ways. In particular, the statement involves integration of a leading order class over a finite cycle in LM, not over all of LM, so the nonrigorous localization step in the formal proof is sidestepped. It should be emphasized that this is only a restatement and not a loop space proof of the index theorem, as the S^1-index theorem is used in the restatement. Along the way, we construct equivariant characteristic forms on LM, such as the equivariant $\hat{A}$-genus and Chern character, which restrict to the corresponding forms on M sitting inside LM as constant loops (Theorem 3.3). It is unclear if the Chern character form we construct is the same as those constructed in [10,11].

In Section 4 we apply similar techniques to the moduli space of pseudoholomorphic maps from a Riemann surface Σ to a symplectic manifold M. We prove that certain Gromov–Witten invariants and gravitational descendants can be expressed in terms of leading order classes and string classes, and we recover the Dilaton Axiom. These techniques work when the GW invariants are really given by integrals over the smooth interior of the compactified moduli space, for which we rely on [12]. In particular, we have to restrict ourselves to genus zero GW invariants for semipositive manifolds. The main geometric observation is that the fibration of (interiors of) moduli spaces associated to forgetting a marked point is flat, so that leading order classes are defined. The main results (Theorems 4.1, 4.6) involve a mixture of string and leading order classes.

In Section 5 we prove that the real cohomology of a based loop group ΩG, for G compact, is generated by leading order Chern–Simons classes. This amounts to noting that the cohomology of G is generated by Chern–Simons classes, and then relating these finite rank classes to the leading order classes. We note that the generators of ΩG can also be written in terms of string classes, a known result [1], and we specifically relate the string and leading order classes (Theorem 5.5). Related results are in [13].

In Section 6 we study leading order classes associated to the gauge theory fibration $\mathcal{A}^* \to \mathcal{A}^*/\mathcal{G}$. This fibration has a natural gauge connection [14,15], whose curvature involves nonlocal Green's operators. Leading order classes only deal with the locally defined symbol of these operators, so the calculation of these classes is relatively easy. In Proposition 6.2, we show that the canonical representative of Donaldson's ν-class [16, Chapter V] in the cohomology of the moduli space $ASD/\mathcal{G}$ of ASD connections on a 4-manifold is the restriction of a leading order form on all of $\mathcal{A}^*/\mathcal{G}$. Thus the ν-class gives information on the cohomology of $\mathcal{A}^*/\mathcal{G}$. It is desirable to extend this construction to cover the more important μ-classes, but this seems to require a theory of leading order currents. We give a preliminary result in this direction.

We would like to thank Michael Murray and Raymond Vozzo for helpful conversations about Section 5, and the referee for insightful comments, particularly about the averaging map in Section 2.

2. Two types of characteristic classes

Perhaps the simplest type of infinite rank vector bundles come from fibrations. Let $Z \to M \stackrel{\pi}{\to} B$ be a locally trivial fibration, with Z, M, B smooth, closed, oriented manifolds, and let $E \to M$ be a smooth bundle. The pushdown bundle $\mathcal{E} = \pi_* E$ is a bundle over B with fiber $\Gamma(E|_{\pi^{-1}(b)})$ over $b \in B$. To specify the topology of $\mathcal{E}$, we can choose either a Sobolev class of H^s sections for the fibers or the Fréchet topology on smooth sections.

Using the transition functions of E, we can check that $\mathcal{E}$ is a smooth bundle with Banach spaces or Fréchet spaces as fibers in these two cases. For local triviality, take a connection ∇ on E, and fix a neighborhood U containing b over which the fibration is trivial. We can assume that U is filled out by radial curves centered at b. Fix a connection D for the fibration, *i.e.*, a complement to the kernel of π_* in TM. For $m \in M_b = \pi^{-1}(b)$, each radial curve has a unique horizontal lift to a curve in M starting at m. For $s \in \Gamma(E|_{\pi^{-1}(b)})$, take the ∇-parallel translation of s along each horizontal lift at m. This gives a smooth isomorphism of $\mathcal{E}_b$ with $\mathcal{E}_{b'}$ for all $b' \in U$.

The connection ∇ pushes down to a connection $\pi_*\nabla = \nabla'$ on $\mathcal{E}$ by

$$\pi_*\nabla_X(s')(m) = \nabla_{X^h}(\tilde{s})(m), \tag{2.1}$$

where X^h is the D-horizontal lift of $X \in T_bB$ to T_mM, $s' \in \Gamma(\mathcal{E})$, and $\tilde{s} \in \Gamma(E)$ is defined by $\tilde{s}(m) = s'(\pi(m))(m)$. Thus ∇' acts as a first order operator on $\mathcal{E}_b$. The curvature Ω' of ∇', defined by

$$\Omega'(X, Y) = \pi_*\nabla_X\pi_*\nabla_Y - \pi_*\nabla_Y\pi_*\nabla_X - \pi_*\nabla_{[X,Y]}$$