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Vector fields with a non-degenerate source

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ABSTRACT

We discuss the solution theory of operators of the form $\nabla_X + A$, acting on smooth sections of a vector bundle with connection ∇ over a manifold M, where X is a vector field having a critical point with positive linearization at some point $p \in M$. As an operator on a suitable space of smooth sections $\Gamma^\infty(U, \mathcal{V})$, it fulfills a Fredholm alternative, and the same is true for the adjoint operator. Furthermore, we show that the solutions depend smoothly on the data ∇, X and A.

1. Introduction

Let M be a manifold of dimension n and X be a smooth vector field on M. We consider points $p \in M$, where the vector field has what we will call a *strictly positive source*, meaning that $X(p) = 0$ and all eigenvalues of the linearization of X at p have a strictly positive real part.

Let furthermore $\mathcal{V}$ be a real or complex vector bundle over M endowed with a connection ∇ and some given endomorphism field A, i.e. a smooth section of the bundle $\mathrm{End}(\mathcal{V})$. In this paper, we discuss properties of the differential operator $\nabla_X + A$, where we assume that X has a strictly positive source at $p \in M$, as explained above. The goal is to solve differential equations of the form

$$(\nabla_X + A)u = \lambda u + v. \tag{1.1}$$

In the case that $M \subseteq \mathbb{R}^n$ is open and A is just a matrix-valued function, we may assume that $p = 0$ and the eigenvalue equation $(\nabla_X + A)u = v$ is equivalent to the system

$$X^i(y)\frac{\partial}{\partial y^i}u^k(y) + A^k_j(y)u^j(y) = v^k(y), \quad k = 1, \ldots, m \tag{1.2}$$

of scalar first order equations, where the functions X^i vanish at zero and the real part of each of the eigenvalues of the matrix $(\mathrm{D}_j X^i)_{ij}$ is positive.

Usually, first order equations can be easily solved with the method of characteristics, but the singular nature of the operator does not admit this approach near the critical point of X. In fact, we will see that operators of this type have some striking analogies to elliptic operators; we show in particular that as an operator on a suitable space of smooth sections

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$\Gamma^\infty(U, \mathcal{V})$, the operator fulfills a Fredholm alternative, and the same is true for the dual operator on the space of distributions $\mathcal{E}'(U, \mathcal{V}^*)$. Furthermore, the solutions depend smoothly on the data and the solution operator is a smooth map between suitable Fréchet spaces.

Operators of the above form appear in various situations in geometry and mathematical physics. For example, the so-called recursive transport equations that appear in the construction of the asymptotic expansion of the heat kernel on a Riemannian manifold or the Hadamard solution associated to a d'Alembert operator on a Lorentzian manifold are of the form (1.1). Also, the transport equations one has to solve in semiclassical analysis when constructing formal WKB expansions to Schrödinger operators $\hbar^2 \Delta + V$ near critical points of the potential V have the form (1.1). We discuss these examples in Section 8.

The study of differential equations of this form has some history in the theory of WKB approximations in semiclassical analysis (see e.g. [1–3]). Most of the results of this paper are therefore not particularly new but presented in a more are general and conceptual form, involving in particular a vector-valued geometric setting. In particular the formulation of Theorem 2.3 seems more clear and less *ad hoc* than the corresponding statements available in the literature. The use of the more refined estimates from Theorem 3.2 to get smooth dependence on initial data (Theorem 7.1) seems to be new.

We use many of the ideas from the cited references, but we will need to adapt the proofs to fit the more general setting we shall discuss.

2. Outline of the results

Definition 2.1. Let M be a manifold of dimension n and X be a smooth vector field on M. A point $p \in M$ with $X(p) = 0$ is called *strictly positive source* of X if the real parts of all eigenvalues of the linearization $\nabla X|_p \in \mathrm{End}(T_pM)$ are strictly positive.

Whenever we speak of eigenvalues of an endomorphism of a finite-dimensional vector space, we always mean the roots of the characteristic polynomial, counted with algebraic multiplicity. In the definition above, ∇ is any connection on TM; because $X(p) = 0$, the linearization $\nabla X|_p$ is independent of the choice of connection.

Definition 2.2. If X has a strictly positive source at $p \in M$, an open neighborhood U of p is called *star-shaped* around p with respect to X, if for all $q \in U$, the flow $\Phi_t(q)$ of X exists for all $t \leq 0$ with

$$\lim_{t \longrightarrow -\infty} \Phi_t(q) = p,$$

and furthermore $\Phi_t(U) \subseteq U$ for all $t \leq 0$. The stable manifold theorem (see for example [4, p. 116]) guarantees the existence of star-shaped neighborhoods around p.

For open subsets $U \subseteq M$, we equip the space of sections $\Gamma^\infty(U, \mathcal{V})$ with its Fréchet topology (induced by the C^m norms on compact subsets of U). The dual space is then $\mathcal{E}'(U, \mathcal{V}^*)$, the space of compactly supported distributions with values in $\mathcal{V}^*$.

Theorem 2.3 (*Fredholm Alternative*). *Let X be a vector field on M with a strictly positive source at p and let U be star-shaped around p with respect to X. Consider the operator $\nabla_X + A$ as a bounded linear operator on $\Gamma^\infty(U, \mathcal{V})$, as well as the dual operator $(\nabla_X + A)'$ on the space $\mathcal{E}'(U, \mathcal{V}^*)$.*

Then either

(a) *λ is not an eigenvalue of $\nabla_X + A$ and the inhomogeneous equation*

$$(\nabla_X + A)u = \lambda u + v \tag{2.1}$$

has exactly one solution for each $v \in \Gamma^\infty(U, \mathcal{V})$; or

(b) *both the homogeneous equation*

$$(\nabla_X + A)u = \lambda u, \quad u \in \Gamma^\infty(U, \mathcal{V}) \tag{2.2}$$

and the dual equation

$$(\nabla_X + A)'T = \lambda T, \quad T \in \mathcal{E}'(U, \mathcal{V}^*) \tag{2.3}$$

have $k < \infty$ linearly independent solutions. Then the inhomogeneous equation (2.1) has a solution if and only if $v \in \ker((\nabla_X + A)' - \lambda)_\perp$. In this case, the space of solutions is k-dimensional affine subspace of $\Gamma^\infty(U, \mathcal{V})$, the direction of which is the space of solutions to (2.2).

In the theorem,

$$\ker((\nabla_X + A)' - \lambda)_\perp := \{u \in \Gamma^\infty(U, \mathcal{V}) \mid T(u) = 0 \, \forall T \in \ker((\nabla_X + A)' - \lambda)\}.$$

We will give a proof in Section 6. A similar theorem holds for the dual operator $(\nabla_X + A)'$, compare Corollary 6.7. Results similar to Theorem 2.3 appear in [3] and partly in [2, Theorem 2.3.1], but in a different form.

In this sense, operators of the form $\nabla_X + A$ behave similar to elliptic operators. There is no analog to elliptic regularity however, and indeed there may be additional non-smooth solutions of (2.2) (see Example 8.1). For this reason, it is suitable to consider the spaces of smooth sections as opposed to some Banach or Hilbert space setting one usually considers.