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Stochastic stability of measures in gradient systems

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HIGHLIGHTS

- Stochastic stability of invariant measures is investigated for gradient systems.
- Noise vanishing limits of Gibbs measures are discussed.
- Examples of non-convergence or instability are given for both additive and multiplicative noise perturbations.

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ABSTRACT

Stochastic stability of a compact invariant set of a finite dimensional, dissipative system is studied in our recent work "Concentration and limit behaviors of stationary measures" (Huang et al., 2015) for general white noise perturbations. In particular, it is shown under some Lyapunov conditions that the global attractor of the systems is always stable under general noise perturbations and any strong local attractor in it can be stabilized by a particular family of noise perturbations. Nevertheless, not much is known about the stochastic stability of an invariant measure in such a system. In this paper, we will study the issue of stochastic stability of invariant measures with respect to a finite dimensional, dissipative gradient system with potential function f. As we will show, a special property of such a system is that it is the set of equilibria which is stable under general noise perturbations and the set S_f of global minimal points of f which is stable under additive noise perturbations. For stochastic stability of invariant measures in such a system, we will characterize two cases of f, one corresponding to the case of finite S_f and the other one corresponding to the case when S_f is of positive Lebesgue measure, such that either some combined Dirac measures or the normalized Lebesgue measure on S_f is stable under additive noise perturbations. However, we will show by constructing an example that such measure stability can fail even in the simplest situation, i.e., in 1-dimension there exists a potential function f such that S_f consists of merely two points but no invariant measure of the corresponding gradient system is stable under additive noise perturbations. Crucial roles played by multiplicative and additive noise perturbations to the measure stability of a gradient system will also be discussed. In particular, the nature of instabilities of the normalized Lebesgue measure on S_f under multiplicative noise perturbations will be exhibited by an example.

1. Introduction

Consider a system of ordinary differential equations

$$\dot{x} = V(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $V = (V^i) \in C(\mathbb{R}^n, \mathbb{R}^n)$. Adding multiplicative including additive white noise $G(x)\dot{W}$ to (1.3), we obtain the following Itô

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stochastic differential equations

$$\mathrm{d}x = V(x)\mathrm{d}t + G(x)\mathrm{d}W, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where W is a standard m-dimensional Brownian motion for some integer $m \geq n$ and $G = (g^{ij})_{n\times m}$ is a matrix-valued function on $\mathbb{R}^n$.

Regarded as a physical model, system (1.1) is often subject to white noise perturbations either from its surrounding environment or from intrinsic uncertainties of a coupling system due to high complexity, large degree of freedom, lack of full knowledge of mechanisms, the need for organizing a large amount of data, etc. Suppose that (1.1) generates a local flow on $\mathbb{R}^n$. Analyzing the impact of noise perturbations on the dynamics of the system is a fundamental issue with respect to both modeling and dynamics. The study of this fundamental issue from a distribution point of view naturally gives rise to the analysis of limit behaviors of stationary measures of the Fokker–Planck equations associated with (1.2) as $G \to 0$ under an appropriate topology.

More precisely, consider noise coefficient matrices lying in the class

$$\tilde{\mathcal{G}} = \{G = (g^{ij}) : \mathrm{Rank}(G) \equiv n,\ g^{ij} \in W^{1,2p}_{\mathrm{loc}}(\mathbb{R}^n),\\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}$$

for some fixed $p > n$. The class $\tilde{\mathcal{G}}$ gives rise to the following class of diffusion matrices:

$$\tilde{\mathcal{A}} = \left\{A = (a^{ij}) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, GL(n, \mathbb{R})) : A = \frac{GG^\top}{2}\right.\\ \left.\text{for some } G \in \tilde{\mathcal{G}}\right\}.$$

For each $A = \frac{GG^\top}{2} = (a^{ij}) \in \tilde{\mathcal{A}}$, the stationary process generated by (1.2) is described by its corresponding stationary measures $\{\mu_A\}$ which are measure-valued solutions of the stationary Fokker–Planck equation associated with (1.2) (see Section 2.1 for details).

Let $\tilde{\mathcal{A}}$ be equipped with the uniform topology of $C(\mathbb{R}^n)$ and consider an *admissible null family* $\mathcal{A} = \{A_\alpha\} \subset \tilde{\mathcal{A}}$, i.e., $\mathcal{A}$ is a directed net with $A_\alpha \to 0$, and the Fokker–Planck equation corresponding to each $A_\alpha \in \mathcal{A}$ admits a stationary measure. Stability of dynamics of (1.1) under the noise family $\mathcal{A}$ can be characterized by the behaviors of $\mathcal{A}$*-limit measures*, i.e., sequential limit points of all stationary measures $\{\mu_{A_\alpha}\}$ of the Fokker–Planck equations corresponding to $\mathcal{A} = \{A_\alpha\}$, as $A_\alpha \to 0$, in the space $M(\mathbb{R}^n)$ of Borel probability measures on $\mathbb{R}^n$ endowed with the weak*-topology. We recall from [1] that a compact invariant set Ω of (1.1) is said to be $\mathcal{A}$*-stable* if for any $\epsilon > 0$ and any open neighborhood W of Ω there exists a $\delta > 0$ such that $\mu_{A_\alpha}(\mathbb{R}^n \setminus W) < \epsilon$ whenever $|A_\alpha| < \delta$. An invariant measure μ of (1.1) is said to be $\mathcal{A}$*-stable* if any sequence of $\{\mu_{A_\alpha}\}$ converges to μ in $M(\mathbb{R}^n)$ as $A_\alpha \to 0$, i.e., μ is the only $\mathcal{A}$-limit measure.

$\mathcal{A}$-stability of a compact invariant set of (1.1) has been extensively investigated in [1]. In particular, it is shown in [1] that if (1.1) admits a Lyapunov function whose second derivatives are bounded, then its global attractor is $\mathcal{A}$-stable with respect to any null family $\mathcal{A}$, and moreover, if the global attractor contains a strong local attractor then there is an admissible null family $\mathcal{A}$ such that the local attractor is $\mathcal{A}$-stable. To the contrary, if the global attractor contains a strong local repeller, then there is an admissible null family $\mathcal{A}$ such that the local repeller is *strongly* $\mathcal{A}$*-unstable*, i.e., no $\mathcal{A}$-limit measure can be concentrated on the repeller. Moreover, if the repeller is a so-called strongly repelling equilibrium, then it is strongly $\mathcal{A}$-unstable with respect to any so-called normal null family $\mathcal{A}$.

In contrast to the case of compact invariant sets, not much is known about the $\mathcal{A}$-stability of invariant measures of (1.1). Of course, if an $\mathcal{A}$-stable compact invariant set is uniquely ergodic, then it is clear that the unique invariant measure is also $\mathcal{A}$-stable. In general, stochastic stability of a non-ergodic invariant measure is much harder to be characterized. We mention some well-known studies in this regard for flows on a 2-torus [2], flows on a circle [3], flows whose ω-limit sets consist of a finite number of fixed points and periodic orbits [4], and flows on a compact manifold admitting SRB measures ([5], see also [6,7] for the case of random perturbations of maps on a compact manifold).

In this paper, we pay particular attention to the stochastic stability of compact invariant sets and invariant measures of a gradient system

$$\dot{x} = -\nabla f(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $f \in C^2(\mathbb{R}^n)$.

As to be seen in this paper, not only does the stochastic stability of compact invariant sets of (1.3) has very special natures, but also the stochastic stability of invariant measures of (1.3) can be characterized in various situations. Besides analyzing the impact of noise perturbations on a gradient system, the study of stochastic stability of invariant measures in a gradient system is closely related to the problem of ergodicity when taking the thermodynamic limit in a huge particle system [8] and the problem of noise stabilizing a multi-stable gradient system [9].

It is well-known that if the gradient system (1.3) admits a weak Lyapunov function then it always generates a positive semiflow in $\mathbb{R}^n$. Another special property of a gradient system is that when it is dissipative, its global attractor is typically simple by consisting of equilibria together with connecting orbits among them. However, it follows from general results of [1] that noise perturbations can remove all the connecting orbits among the equilibria in the global attractor of a dissipative gradient system. More precisely, if (1.3) admits a C^2 Lyapunov function whose second derivative is bounded, then (a) the set E of critical points of f is $\mathcal{A}$-stable with respect to any null family $\mathcal{A}$; (b) any finite set $\mathcal{J}_0$ (resp. $\mathcal{R}_0$) of isolated local minimal (maximal) points of f is $\mathcal{A}$-stable (resp. strongly $\mathcal{A}$-unstable) with respect to a particular null family $\mathcal{A}$; (c) any simple local maximal point of f is strongly $\mathcal{A}$-unstable with respect to any normal null family $\mathcal{A}$ (see Theorem 2.1 for details). When either the set E or $\mathcal{J}_0$ is a singleton, the $\mathcal{A}$-stability of the corresponding Dirac measure obviously follows from that of the E or $\mathcal{J}_0$ with respect to all or a particular null family $\mathcal{A}$.

However, stochastic stability of (1.3) has very special natures under the additive white noise perturbation $\sqrt{2\epsilon}\dot{W}$ to (1.3), where $\epsilon > 0$ is a small parameter. Under the condition that

(H) there are positive constants R and β such that $f(x) \geq \beta \log |x|$ for all $|x| \geq R$,

the Fokker–Planck equation corresponding to

$$\mathrm{d}x = -\nabla f(x)\mathrm{d}t + \sqrt{2\epsilon}\mathrm{d}W, \quad x \in \mathbb{R}^n \tag{1.4}$$

for each $\epsilon > 0$ admits a stationary measure μ_ϵ, called *Gibbs measure*, with density

$$u^\epsilon(x) = k_\epsilon \mathrm{e}^{-\frac{f(x)}{\epsilon}}, \quad x \in \mathbb{R}^n,$$

called *Gibbs state*, where

$$k_\epsilon = \frac{1}{\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{f(x)}{\epsilon}}\mathrm{d}x}.$$

Limit behaviors of (continuum) Gibbs measures have been explicitly investigated in many situations (see, e.g., [10,11]). These limit behaviors lead to various $\mathcal{A}_0$-stability results of (1.3), where $\mathcal{A}_0$ denote the family of diffusion matrices $\{\epsilon I\}$ corresponding to the additive noise perturbations. More precisely, it follows from the limit characterizations of Gibbs measures in [11] that if the condition (H) holds, then the set S_f of all global minimal points