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On the spectrum of the Dirichlet-to-Neumann operator acting on forms of a Euclidean domain

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ABSTRACT

We compute the whole spectrum of the Dirichlet-to-Neumann operator acting on differential p-forms on the unit Euclidean ball. Then, we prove a new upper bound for its first eigenvalue on a domain Ω in Euclidean space in terms of the isoperimetric ratio $\mathrm{Vol}(\partial\Omega)/\mathrm{Vol}(\Omega)$.

1. Introduction

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional compact and connected Riemannian manifold with smooth boundary Σ. The Dirichlet-to-Neumann operator on functions associates, to each function defined on the boundary, the normal derivative of its harmonic extension to Ω. More precisely, if $f \in C^\infty(\Sigma)$, its harmonic extension $\widehat{f}$ is the unique smooth function on Ω satisfying

$$\begin{cases} \Delta\widehat{f} = 0 & \text{in } \Omega, \\ \widehat{f} = f & \text{on } \Sigma \end{cases}$$

and the Dirichlet-to-Neumann operator $T^{[0]}$ is defined by:

$$T^{[0]}f := -\frac{\partial\widehat{f}}{\partial N}$$

where N is the inner unit normal to Σ. It is a well known result (see [1] for example) that $T^{[0]}$ is a first order elliptic, non-negative and self-adjoint pseudo-differential operator with discrete spectrum

$$0 = \nu_{1,0}(\Omega) < \nu_{2,0}(\Omega) \leq \nu_{3,0}(\Omega) \leq \cdots \nearrow \infty.$$

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As Ω is connected, $\nu_{1,0}(\Omega) = 0$ is simple, and its eigenspace consists of the constant functions. The first positive eigenvalue has the following variational characterization:

$$\nu_{2,0}(\Omega) = \inf\left\{ \frac{\int_\Omega |df|^2}{\int_\Sigma f^2} : f \in C^\infty(\Omega)\setminus\{0\}, \int_\Sigma f = 0 \right\}. \quad (1)$$

The study of the spectrum of $T^{[0]}$ was initiated by Steklov in [2]. We note that the Dirichlet-to-Neumann map is closely related to the problem of determining a complete Riemannian manifold with boundary from the Cauchy data of harmonic functions. Indeed, a striking result of Lassas, Taylor and Uhlmann [3] states that if the manifold Ω is real analytic and has dimension at least 3, then the knowledge of $T^{[0]}$ determines Ω up to isometry.

It can be easily seen that the eigenvalues of the Dirichlet-to-Neumann map of the unit ball $\mathbf{B}^{n+1}$ in $\mathbf{R}^{n+1}$ are $\nu_{k,0} = k$, with $k = 0, 1, 2, \ldots$ and the corresponding eigenspace is given by the vector space of homogeneous harmonic polynomials of degree k restricted to the sphere $\partial\mathbf{B}^{n+1}$.

1.1. The Dirichlet-to-Neumann operator on forms

In [4], we extend the definition of the Dirichlet-to-Neumann map $T^{[0]}$ acting on functions to an operator $T^{[p]}$ acting on $\Lambda^p(\Sigma)$, the vector bundle of differential p-forms of $\Sigma = \partial\Omega$ for $0 \le p \le n$. This is done as follows. Let ω be a form of degree p on Σ, with $p = 0, 1, \ldots, n$. Then there exists a unique p-form $\widehat{\omega}$ on Ω such that:

$$\begin{cases} \Delta\widehat{\omega} = 0 \\ J^\star\widehat{\omega} = \omega, \quad i_N\widehat{\omega} = 0. \end{cases}$$

Here $\Delta = d\delta + \delta d$ is the Hodge Laplacian acting on $\Lambda^p(\Omega)$ (the bundle of p-forms on Ω) $J^\star : \Lambda^p(\Omega) \to \Lambda^p(\Sigma)$ is the restriction map and i_N is the interior product of $\widehat{\omega}$ with the inner unit normal vector field N. The existence and uniqueness of the form $\widehat{\omega}$ (called the *harmonic tangential extension of* ω) is proved, for example, in Schwarz [5]. We let:

$$T^{[p]}\omega = -i_N d\widehat{\omega}.$$

Then $T^{[p]} : \Lambda^p(\Sigma) \to \Lambda^p(\Sigma)$ defines a linear operator, *the (absolute) Dirichlet-to-Neumann operator*, which reduces to the classical Dirichlet-to-Neumann operator $T^{[0]}$ acting on functions when $p = 0$. We proved in [4] that $T^{[p]}$ is an elliptic self-adjoint and non-negative pseudo-differential operator, with discrete spectrum

$$0 \le \nu_{1,p}(\Omega) \le \nu_{2,p}(\Omega) \le \cdots$$

tending to infinity. Note that $\nu_{1,p}(\Omega)$ can in fact be zero: it is not difficult to prove that $\mathrm{Ker}T^{[p]}$ is isomorphic to $H^p(\Omega)$, the p-th absolute de Rham cohomology space of Ω with real coefficients.

The operator $T^{[p]}$ belongs to a family of operators first considered by G. Carron in [6]. Other Dirichlet-to-Neumann operators acting on differential forms, but different from ours, were introduced by Joshi and Lionheart in [7], and Belishev and Sharafutdinov in [8]. In fact, our operator $T^{[p]}$ appears in a certain matrix decomposition of the Joshi and Lionheart operator (see [4] for complete references). However, one advantage of our operator is its self-adjointness, which permits to study its spectral and variational properties. In particular one has (see [4]):

$$\nu_{1,p}(\Omega) = \inf\left\{ \frac{\int_\Omega |d\omega|^2 + |\delta\omega|^2}{\int_\Sigma |\omega|^2} : \omega \in \Lambda^p(\Omega)\setminus\{0\},\ i_N\omega = 0 \text{ on } \Sigma \right\}. \quad (2)$$

For $p = 0, \ldots, n$, we also have a dual operator $T_D^{[p]} : \Lambda^p(\Omega) \to \Lambda^p(\Omega)$ with eigenvalues $\nu^D_{k,p}(\Omega) = \nu_{k,n-p}(\Omega)$ (for its definition, we refer to [4]). Here we just want to observe that:

$$\nu^D_{1,p}(\Omega) = \inf\left\{ \frac{\int_\Omega |d\omega|^2 + |\delta\omega|^2}{\int_\Sigma |\omega|^2} : \omega \in \Lambda^{p+1}(\Omega)\setminus\{0\}, J^\star\omega = 0 \text{ on } \Sigma \right\}. \quad (3)$$

In [4], we obtained sharp upper and lower bounds of $\nu_{1,p}(\Omega)$ in terms of the extrinsic geometry of its boundary: let us briefly explain the main lower bound.

Fix $x \in \Sigma$ and consider the principal curvatures $\eta_1(x), \ldots, \eta_n(x)$ of Σ at x; if $p = 1, \ldots, n$ and $1 \le j_1 < \cdots < j_p \le n$ is a multi-index, we call the number $\eta_{j_1}(x) + \cdots + \eta_{j_p}(x)$ a *p-curvature* of Σ. We set:

$$\sigma_p(x) = \inf\{\eta_{j_1}(x) + \cdots + \eta_{j_p}(x) : 1 \le j_1 < \cdots < j_p \le n\}$$
$$\sigma_p(\Sigma) = \inf\{\sigma_p(x) : x \in \Sigma\}$$

and say that Σ is *p-convex* if $\sigma_p(\Sigma) \ge 0$ that is, if all p-curvatures of Σ are non-negative. For example Σ is 1-convex if and only if it is convex in the usual sense, and it is n-convex if and only if it is mean-convex (that is, it has non-negative mean curvature everywhere).