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Endomotives of toric varieties

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ABSTRACT

We construct endomotives associated to toric varieties, in terms of the decomposition of a toric variety into torus orbits and the action of a semigroup of toric morphisms. We show that the endomotives can be endowed with time evolutions and we discuss the resulting quantum statistical mechanical systems. We show that in particular, one can construct a time evolution related to the logarithmic height function. We discuss relations to $\mathbb{F}_1$-geometry.

1. Introduction

The notion of endomotive was introduced in [1] as a way to describe, in terms of arithmetic data, the construction of quantum statistical mechanical systems associated to number theory, starting with the prototype Bost–Connes system [2]. Endomotives are algebras obtained from projective limits of Artin motives (zero dimensional algebraic varieties) endowed with semigroup actions. Out of these algebraic data one obtains noncommutative spaces, in the form of semigroup crossed product C^*-algebras.

There are natural time evolutions on the C^*-algebra of an endomotive, and the associated Hamiltonian, partition function and KMS equilibrium states relate to properties of L-functions in cases arising from number theory, see [1,3,4]. In particular, endomotives were recently used by Bora Yalkinoglu to construct arithmetic subalgebras for all the quantum statistical mechanics associated to number fields, [4], in the context of the noncommutative geometry approach to the explicit class field theory problem. It is also known, see [5–8,4], that the notion of endomotive is closely related to the notion of Λ-rings studied by Borger, in his approach to geometry over the "field with one element" $\mathbb{F}_1$.

An interesting question in the theory of endomotives is whether the construction extends to more general algebro-geometric objects, besides the cases underlying the construction of quantum statistical mechanical systems of number fields, that was the main focus in [1,3,4]. Toric varieties are a natural choice of a class of varieties for which this question can be addressed. In fact, toric varieties constitute an important class of algebraic varieties, which is sufficiently concrete and well understood [9,10] to provide a good testing ground for various constructions. Moreover, toric varieties play an important role in the theory of $\mathbb{F}_1$-geometry. A first step in the direction of the construction of associated endomotives already existed, in the form of the multivariable Bost–Connes systems introduced in [8], which we will interpret here as the simplest case of our construction for toric varieties, corresponding to the case where the variety is just a torus $\mathbb{T}^n$.

In this paper, we construct endomotives for abstract toric varieties, generalizing the existing constructions of the Bost–Connes system and its generalizations.

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In the rest of this section, we recall the basic notion of an endomotive, as defined in [1] and the main properties of the associated quantum statistical mechanical systems. We also recall the torus-cone correspondence for toric varieties, which will be the basis of our construction. We then identify a semigroup of toric morphisms which we will use in the endomotive data.

In Section 2 we describe the construction of the endomotives of abstract toric varieties. We give two variants of the construction, which we refer to as the additive and multiplicative case, which correspond, respectively, to the abelian part of the endomotive algebra being a direct sum or a tensor product of contributions from the single orbits. The direct sum choice is more closely tied up to the geometry of the variety, as it corresponds to the decomposition into a union of orbits, with the abelian part of the endomotive given by the algebra of functions on a set of algebraic points on the toric variety obtained as a projective limit over the action of the semigroup. The tensor product case corresponds instead to regarding the torus orbits as defining independent quantum mechanical systems, so that the resulting partition function will decompose as a product over orbits. In both cases we obtain Hilbert space representations of the abelian algebras and of the semigroup, in such a way that they determine a representation of the semigroup crossed product algebra. We describe the explicit generators and relations of the crossed product. We then describe a general procedure to construct time evolutions on the algebra with the corresponding Hamiltonians that are the infinitesimal generators in the given representation. We describe the group of symmetries and the partition function. In some especially nice cases, from the point of view of symmetries of the fan defining the toric variety, we give a more concrete description of the Hamiltonian and the partition function.

In Section 3, we relate the endomotives of toric varieties to $\mathbb{F}_1$-geometry, both in the sense of Borger [5], via the notion of Λ-ring, and in the sense of Soulé [11]. We also show that a weaker form of the endomotive construction can be extended from toric varieties to torified spaces in the sense of the approach to $\mathbb{F}_1$-geometry of López-Peña and Lorscheid [12].

In Section 4, we focus on the case of projective toric varieties, and in particular on the concrete example of projective spaces. We replace a set of distinguished points in the torus orbits used in the abstract construction of Section 2 with the set of $\bar{\mathbb{Q}}$-algebraic points of the variety with bounded height and degree over $\mathbb{Q}$, and we describe a time evolution and covariant representations of the resulting C^*-dynamical system related to the logarithmic height function. We show a variant of the construction for the case of affine spaces.

Finally, in Section 5, we discuss briefly the Gibbs equilibrium states for the quantum statistical mechanical systems of endomotives of abstract toric varieties.

1.1. The notion of endomotive

We recall here briefly the notion of endomotive from [1] and the main properties we will be discussing in the rest of the paper.

The data of an endomotive consist of a projective system X_α of zero dimensional algebraic varieties over a field $\mathbb{K}$, where $X_\alpha = \mathrm{Spec}(A_\alpha)$, together with an action by endomorphisms of a semigroup S on the limit $X = \varprojlim_\alpha X_\alpha$. In [1] the field $\mathbb{K}$ is assumed to be a number field.

In [1] the semigroup S is assumed to be countably generated and *abelian*, while more general situations with S not necessarily abelian were discussed, for instance, in [8] and will also be considered here.

At the algebraic level, one considers the algebraic semigroup crossed product $\mathbb{K}$-algebra $A \rtimes S$, where $A = \varinjlim_\alpha A_\alpha$ and $X = \mathrm{Spec}(A)$. This is generated algebraically by elements $a \in A$ and additional generators μ_s and μ_s^*, for $s \in S$, satisfying the relations $\mu_s^* \mu_s = 1$, $\mu_s \mu_s^* = \phi_s(1)$, where ϕ_s is the endomorphism of A corresponding to $s \in S$, and $\mu_{s_1 s_2} = \mu_{s_1} \mu_{s_2}$, $\mu_{s_2 s_1}^* = \mu_{s_1}^* \mu_{s_2}^*$, $\mu_s\, a = \phi_s(a)\, \mu_s$, and $a\, \mu_s^* = \mu_s^* \phi_s(a)$, for all $s, s_1, s_2 \in S$ and for all $a \in A$.

1.1.1. Quantum statistical mechanical systems of endomotives

At the analytic level, one considers the C^*-algebra $\mathcal{A} = C(X(\bar{\mathbb{K}})) \rtimes S$, where the set $X(\bar{\mathbb{K}})$ is topologized with the profinite topology. One regards this as the algebra of observables of a quantum statistical mechanical system, with a time evolution given by a one parameter family of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$.

A covariant representation of the C^*-dynamical system $(\mathcal{A}, \sigma_t)$ is a pair (π, H) of a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the algebra by bounded operators on a Hilbert space $\mathcal{H}$, and an operator H on $\mathcal{H}$ (usually unbounded) with the property that

$$\pi(\sigma_t(a)) = e^{-itH}\, \pi(a)\, e^{itH}. \tag{1.1}$$

One says that H is the Hamiltonian generating the time evolution σ_t in the representation π.

The typical form of the time evolution considered in [1] arises from the modular automorphism group σ_t^φ associated to a state φ determined by a measure on X. Here we will give a construction of time evolutions on endomotives, based more generally on semigroup homomorphisms $g : S \to \mathbb{R}_+^*$ for which there exists an associated function h, with appropriate scaling properties, so that the pair (g, h) determine a time evolution and the corresponding Hamiltonian in an assigned representation (see Proposition 2.8).