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Deformations of generalized holomorphic structures

Yicao Wang

Department of Mathematics, Hohai University, Nanjing 210098, China

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ABSTRACT

A deformation theory of generalized holomorphic structures in the setting of (generalized) principal fibre bundles is developed. It allows the underlying generalized complex structure to vary together with the generalized holomorphic structure. We study the related differential graded Lie algebra, which controls the deformation problem via the Maurer–Cartan equation. As examples, we check the content of the Maurer–Cartan equation in detail in the special cases where the underlying generalized complex structure is symplectic or complex. A deformation theorem, together with some non-obstructed examples, is also included.

1. Introduction

In generalized complex geometry, the notion of generalized holomorphic structures is the analogue of holomorphic structures in classical complex geometry, including flat bundles over symplectic manifolds, co-Higgs bundles and holomorphic Poisson modules as extreme examples. These examples are also the most studied cases to date: the flat case is the most trivial one; N. Hitchin has studied certain aspects of co-Higgs bundles and also provided some interesting examples [1,2], while [3] contains a detailed investigation of stable co-Higgs bundles over $\mathbb{P}^1$ and $\mathbb{P}^2$; the case of holomorphic Poisson modules is less touched, in particular in the setting of generalized complex geometry—[4,5] contain some topics concerning this. The construction of more general generalized holomorphic structures often involves more effort; for some progress in this direction see [5,6].

In [6,7] the author has explored some local features of generalized holomorphic structures. In the formalism of reduction theory of Courant algebroids and Dirac structures developed in [8], the author has also extended the notion of generalized holomorphic structures to the context of (generalized) principal bundles [6]. This paper is then a continuation of that work, motivated by the attempt to find more examples of generalized holomorphic structures.

One possible way to obtain more examples of generalized holomorphic structures is by deforming a given one. Recall that in classical theory of deformations of holomorphic structures [9], one fixes a compact complex manifold (M, J) together with a holomorphic vector bundle V and tries to find nearby holomorphic structures, *but all these holomorphic structures are w.r.t. the same complex structure J*. Then infinitesimal deformations are contained in $H^1(M, \mathcal{O}(\mathrm{End}(V)))$ while the obstructions for an infinitesimal deformation to be integrable live in $H^2(M, \mathcal{O}(\mathrm{End}(V)))$.

One can certainly routinely apply a similar method to the generalized case, *where the underlying generalized complex structure should be fixed, and in this sense we call the resulting theory traditional.*. However, there are some drawbacks. First,

E-mail addresses: yicwang@mail.ustc.edu.cn, yicwang@hhu.edu.cn.

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when deforming a usual holomorphic structure (viewed as a generalized holomorphic structure), one cannot go too far and at most gets co-Higgs bundles. Second, the relevant cohomology groups (e.g. those associated to a Poisson module) are, to some extent, the starting point to find nearby generalized holomorphic structures, but generally there still lacks any effective way to compute them. Thus, for practical purposes, e.g. to probe more general generalized holomorphic vector bundles, this way of deformation is not that useful.

Therefore, in this paper, we will develop a more general deformation theory, which, more or less, can overcome the above drawbacks. As many existing examples of generalized complex manifolds are obtained by deforming simple ones, our choice is that, we *no longer* restrict ourselves to a *fixed* underlying generalized complex structure—the generalized holomorphic structure varies more freely, because the underlying generalized complex structure is also allowed to vary. In this direction, the formalism of [6] is rather suitable for our purpose, so we start in the context of principal bundles and then work out its vector-bundle counterpart.

The paper is organized as follows. In Section 2, we collect the necessary basics of generalized complex geometry. Section 3 is devoted to finding the correct differential graded Lie algebra (DGLA for short) governing the deformation problem. We show how to define the differential and bracket at the level of equivariant objects over the principal bundle **P** and then descend to the base manifold M. It turns out that the resulting DGLA is an extension of the DGLA controlling deformations of the underlying generalized complex structure by the DGLA controlling traditional deformations of the generalized holomorphic structure. This results in the infinitesimal deformation theory being described by a long exact sequence of cohomology groups (cf. Theorem 3.6). In Section 4 we present the Maurer–Cartan equation and investigate it in detail in the cases where the underlying generalized complex structure is actually symplectic or complex. Some new possibilities occur, which are missing in the existing literature. Section 5 is devoted to proving the deformation theorem (cf. Theorem 5.3). As the procedure is rather standard, we only outline the proof. Examples are presented, in which the obstruction vanishes.

2. Some preliminaries

We collect the basic material concerning generalized complex structures and generalized holomorphic structures. The most relevant references are [4,6,8,10]. In this paper, M will always be a connected orientable smooth $2m$-manifold.

Generalized geometry is the geometry related to the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, or more generally, a so-called exact Courant algebroid E.

Definition 2.1. A Courant algebroid over M is a real vector bundle $E \to M$ with a bracket $[\cdot,\cdot]_c$ (Courant bracket) on $\Gamma(E)$, a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, and an anchor map $\pi : E \to TM$, satisfying the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

- $\pi([e_1, e_2]_c) = [\pi(e_1), \pi(e_2)]$,
- $[e_1, [e_2, e_3]_c]_c = [[e_1, e_2]_c, e_3]_c + [e_2, [e_1, e_3]_c]_c$,
- $[e_1, fe_2]_c = f[e_1, e_2]_c + (\pi(e_1)f)e_2$,
- $\pi(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2]_c, e_3\rangle + \langle e_2, [e_1, e_3]_c\rangle$,
- $[e_1, e_1]_c = \frac{1}{2}\mathcal{D}\langle e_1, e_1\rangle$,

where $\mathcal{D} = \pi^* \circ d : C^\infty(M) \to \Gamma(E)$ (E and E^* are identified using $\langle\cdot,\cdot\rangle$).

Definition 2.2. E is called exact if it is an extension of TM by T^*M, i.e. the sequence

$$0 \longrightarrow T^*M \stackrel{\pi^*}{\longrightarrow} E \stackrel{\pi}{\to} TM \longrightarrow 0$$

is exact.

Courant algebroids encountered in this paper are all exact and called Courant algebroids for short. Given E, one can always find an isotropic right splitting $s : TM \to E$, which has a curvature form $H \in \Omega^3_{cl}(M)$ defined by

$$H(X, Y, Z) = \langle [s(X), s(Y)]_c, s(Z)\rangle, \quad X, Y, Z \in \Gamma(TM).$$

By the bundle isomorphism $s + \pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure can be transported onto $\mathbb{T}M$. Then the pairing $\langle\cdot,\cdot\rangle$ is the natural one, i.e. $\langle X + \xi, Y + \eta\rangle = \xi(Y) + \eta(X)$, and the Courant bracket is

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y\iota_X H,$$

called the H-twisted Courant bracket. Different splittings are related by B-field transforms, i.e. $e^B(X + \xi) = X + \xi + \iota_X B$, where B is a 2-form.

A Courant algebroid E has more symmetries than the tangent bundle; in particular, the left adjoint action by a section of E gives rise to an infinitesimal inner automorphism of E.

An isotropic subbundle $A \subset E$ is called a generalized distribution and called integrable if it is involutive w.r.t. the Courant bracket. An integrable maximal generalized distribution L is called a Dirac structure. These notions can be complexified and what interests us here is the following complex Dirac structure[1]:

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[1] We use $V_{\mathbb{C}}$ to denote the complexification of a real vector space or bundle V.