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Equivariant mean field flow

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ABSTRACT

We consider a gradient flow associated to the mean field equation on (M, g), a compact Riemannian surface without boundary. We prove that this flow exists for all time. Moreover, letting G be a group of isometry acting on (M, g), we obtain the convergence of the flow to a solution of the mean field equation under suitable hypothesis on the orbits of points of M under the action of G.

1. Introduction

Let (M, g) be a compact Riemannian surface without boundary, we will study an evolution problem associated to the mean field equation:

$$\Delta u + \rho \left(\frac{fe^u}{\int_M fe^u dV} - \frac{1}{|M|} \right) = 0, \tag{1.1}$$

where ρ is a real parameter, $|M|$ stands for the volume of M with respect to the metric $g, f \in C^\infty(M)$ is a given function supposed strictly positive and Δ is the Laplacian with respect to the metric g. The mean field equation appears in statistical mechanics from Onsager's vortex model for turbulent Euler flows. More precisely, in this setting, the solution u of the mean field equation is the stream function in the infinite vortex limit (see [1]). This equation is also linked to the study of condensate solutions of the abelian Chern–Simons–Higgs model (see for example [2–5]). Eq. (1.1) is also related to conformal geometry. When (M, g) is the standard sphere and $\rho = 8\pi$, the problem to find a solution to Eq. (1.1) is called the Nirenberg Problem. The geometrical meaning of this problem is that, if u is a solution of (1.1), the conformal metric $e^u g$ admits a Gaussian curvature equal to $\frac{\rho f}{2}$.

Eq. (1.1) is the Euler–Lagrange equation of the nonlinear functional

$$I_\rho(u) = \frac{1}{2} \int_M |\nabla u|^2 dV + \frac{\rho}{|M|} \int_M u dV - \rho \log \left(\int_M fe^u dV \right), \quad u \in H^1(M). \tag{1.2}$$

By using the well-known Moser–Trudinger inequality (see inequality (3.2)), one can easily obtain the existence of solutions of (1.1) for $\rho < 8\pi$ by minimizing I_ρ. The existence of solutions becomes much harder when $\rho \geq 8\pi$. In fact, in this case,

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the functional I_ρ is not coercive. The existence of solutions to Eq. (1.1) has been intensively studied these last decades when $\rho \geq 8\pi$. Many partial existence results have been obtained according to the value of ρ and the topology of M (see for example [6–9] in the references therein). Recently, Djadli [10] proves the existence of solutions to (1.1) for all Riemannian surfaces when $\rho \neq 8k\pi$, $k \in \mathbb{N}^*$, by studying the topology of sublevels $\{I_\rho \leq -C\}$ to achieve a min–max scheme (already introduced in Djadli–Malchiodi [11]).

In this paper, we consider the evolution problem associated to the mean field equation, that is the following equation

$$\begin{cases} \dfrac{\partial}{\partial t}e^u = \Delta u + \rho\left(\dfrac{fe^u}{\int_M fe^u dV} - \dfrac{1}{|M|}\right) \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where $u_0 \in C^{2+\alpha}(M)$, $\alpha \in (0,1)$, is the initial data. It is a gradient flow with respect to the following functional:

$$E_f(u) = \frac{1}{2}\int_M |\nabla u|^2 dV + \frac{\rho}{|M|}\int_M u dV - \rho \ln\left(\int_M fe^u dV\right), \quad u \in H^1(M). \tag{1.4}$$

We first prove the global existence of the flow (1.3). We obtain the following result.

Theorem 1.1. *For all $u_0 \in C^{2+\alpha}(M)$ $(0 < \alpha < 1)$, all $\rho \in \mathbb{R}$ and all functions $f \in C^\infty(M)$ strictly positive, there exists a unique global solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{\mathrm{loc}}(M \times [0,+\infty))$ of (1.3).*

Next, we investigated the convergence of the flow when the initial data and the function f are invariant under an isometry group acting on (M,g). A lot of works have been done for prescribed curvature problems invariant under an isometry group, we refer to [12–15] and the references therein. Before giving a more precise statement of our results, we introduce some notations. Let G be an isometry group of (M,g). For all $x \in M$, we define $O_G(x)$ as the orbit of x under the action of G, i.e.

$$O_G(x) = \{y \in M : y \in \sigma(x),\ \forall \sigma \in G\}.$$

$|O_G(x)|$ will stand for the cardinal of $O_G(x)$. We say that a function $f : M \to \mathbb{R}$ is G-invariant if $f(\sigma(x)) = f(x)$ for all $x \in M$ and $\sigma \in G$. We define $C^\infty_G(M)$ (resp. $C^{2+\alpha}_G(M)$, $\alpha \in (0,1)$) as the space of functions $f \in C^\infty(M)$ (resp. $f \in C^{2+\alpha}_G(M)$) such that f is G-invariant. We prove the convergence of the flow under suitable hypothesis on G allowing us also to handle the critical case when $\rho = 8k\pi$, $k \in \mathbb{N}^*$.

Theorem 1.2. *Let G be an isometry group acting on (M,g) such that*

$$|O_G(x)| > \frac{\rho}{8\pi}, \quad \forall x \in M,$$

and $f \in C^\infty_G(M)$ be a strictly positive function. Then, for all initial $u_0 \in C^{2+\alpha}_G(M)$, the global solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{\mathrm{loc}}(M \times [0,+\infty))$ of (1.3) converges in $H^2(M)$ to a function $u_\infty \in C^\infty_G(M)$ solution of the mean field equation (1.1).

Assuming that f is a positive constant and $G = \mathrm{Isom}(M,g)$, the group of all isometries of (M,g), we obtain the following.

Corollary 1.1. *Suppose that for all $x \in M$, we have $|O_G(x)| = +\infty$ then, for all $\rho \in \mathbb{R}$, the solution of the flow (1.3) converges in $H^2(M)$ to a function $u_\infty \in C^\infty_G(M)$ solution of the mean field equation (1.1).*

Remark 1.1. Taking $M = \mathbb{S}^1 \times \mathbb{S}^1$ endowed with the product metric and $G = \mathrm{Isom}(M,g)$, we have, for all $x \in M$, $|O_G(x)| = +\infty$.

If f is not constant, we also have the following.

Corollary 1.2. *If $\rho < 16\pi$ and $f \in C^\infty(\mathbb{S}^2)$ is an even function then the flow (1.3) converges in $H^2(\mathbb{S}^2)$ to an even function $u_\infty \in C^\infty(\mathbb{S}^2)$ solution of the mean field equation.*

The plan of this paper is the following. In Section 2, we will prove Theorem 1.1. In Section 3, we will give an improved Moser–Trudinger inequality for G-invariant functions. In Section 4, we establish Theorem 1.2: first, using our improved Moser–Trudinger inequality, we obtain a uniform (in time) $H^1(M)$ bound for the solution $u(t)$ of (1.3) where $u(t) : M \to \mathbb{R}$ is defined by $u(t)(x) = u(x,t)$. Then, from the previous estimate, we will derive a uniform $H^2(M)$ bound. Theorem 1.2 will follow from this last estimate.

2. Proof of Theorem 1.1

In this section, we prove the global existence of the flow (1.3). We begin by noticing that, since the flow (1.3) is parabolic, standard methods provide short time existence and uniqueness for it. Thus, there exists $T_1 > 0$ such that $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(M \times [0,T_1])$ is a solution of the flow. It is also easy to see, integrating (1.3) on M, that, for all $t \in [0,T_1]$, we