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Milnor–Selberg zeta functions and zeta regularizations

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ABSTRACT

By a similar idea for the construction of Milnor's gamma functions, we introduce "higher depth determinants" of the Laplacian on a compact Riemann surface of genus greater than one. We prove that, as a generalization of the determinant expression of the Selberg zeta function, this higher depth determinant can be expressed as a product of multiple gamma functions and what we call a Milnor–Selberg zeta function. It is shown that the Milnor–Selberg zeta function admits an analytic continuation, a functional equation and, remarkably, has an Euler product.

1. Introduction

In 1983, Milnor [1] introduced a family of functions $\{\gamma_r(z)\}_{r\geq 1}$ what he thought as a kind of "higher depth gamma functions". These are defined as partial derivatives of the Hurwitz zeta function $\zeta(w,z) := \sum_{n=0}^{\infty}(n+z)^{-w}$ at non-positive integer points with respect to the variable w. We call $\gamma_r(z)$ a Milnor gamma function of depth r and will denote it by $\mathbf{\Gamma}_r(z)$ (see [2] for several analytic properties of $\mathbf{\Gamma}_r(z)$). The purpose of Milnor's study about such functions is to construct functions satisfying a modified version of the Kubert identity $f(x) = m^{s-1}\sum_{k=0}^{m-1} f(\frac{x+k}{m})$ [3], which plays an important role in the study of Iwasawa theory.

The aim of the present paper is, as an analogue of the study of the Milnor gamma functions, to investigate "higher depth determinants" of the Laplacians on compact Riemann surfaces of genus $g \geq 2$ with negative constant curvature. Let us give the definition of the higher depth determinant in more general situations. Let A be a linear operator on some space. We assume that A has only discrete spectrum with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Define a spectral zeta function $\zeta_A(w,z)$ of Hurwitz's type by

$$\zeta_A(w,z) := \sum_{j=0}^{\infty}(\lambda_j + z)^{-w}.$$

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We further assume that the series converges absolutely and uniformly for z on any compact set in some right half w-plane, and can be continued meromorphically to a region containing $w = 1 - r$ for $r \in \mathbb{N}$. Moreover, we assume that it is holomorphic at $w = 1 - r$. In such a situation, we define a *higher depth determinant* of A of depth r by

$$\mathrm{Det}_r(A+z) := \exp\left(-\frac{\partial}{\partial w}\zeta_A(w,z)\Big|_{w=1-r} \right).$$

When $r = 1$, this gives the usual (zeta-regularized) determinant of A. Notice that if A is a finite-rank operator and $\lambda_1, \ldots, \lambda_N$ are its eigenvalues, then we have $\mathrm{Det}_r(A) = \prod_{j=1}^{N} \lambda_j^{\lambda_j^{r-1}}$, whence $\mathrm{Det}_r(A \oplus B) = (\mathrm{Det}_r A) \cdot (\mathrm{Det}_r B)$ if both A and B are finite-rank.

These "higher depth" objects including Milnor's gamma functions are naturally arising from number theoretic problems. Let us provide two such examples. (i) It is known that the associated zeta function of the real analytic Eisenstein series [4] or the prehomogeneous vector space of symmetric matrices [5], for instance, are expressed as polynomials in several shift of the Riemann zeta functions. This means that the corresponding regularized products are given by a product of "higher depth" functions. (ii) As in [6], in order to understand/describe special values of the spectral zeta function for non-commutative harmonic oscillators, we are naturally lead to define the notion of "residual modular forms", which are generalizations of both classical holomorphic modular forms of integral weight and Eichler (automorphic) integrals [7]. It becomes clarified that the differential Eisenstein series, which are defined by derivatives of the generalized Eisenstein series [8] at negative integers, help to obtain a basis of the space of residual modular forms for a congruence subgroup of the modular group.

To state our main results, let us recall the case of the usual determinant of the Laplacian, that is, the case $r = 1$. Let $\mathbb{H}$ be the complex upper half plane with the Poincaré metric and Γ a discrete, co-compact torsion-free subgroup of $SL_2(\mathbb{R})$. Then, $\Gamma\backslash\mathbb{H}$ becomes a compact Riemann surface of genus $g \geq 2$. Let $\Delta_\Gamma = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ be the Laplacian on $\Gamma\backslash\mathbb{H}$, λ_j the jth eigenvalue of Δ_Γ and put $\mathrm{Spec}\,(\Delta_\Gamma) := \{\lambda_j \mid j \in \mathbb{N}_0\}$. We write $\lambda_j = r_j^2 + \frac{1}{4}$ where $r_j \in i\mathbb{R}_{>0}$ if $0 \leq \lambda_j < \frac{1}{4}$ and $r_j \geq 0$ otherwise. Moreover, let $\alpha_j^{\pm} := \frac{1}{2} \pm i r_j$. Notice that $\alpha_j^{\pm} \in [0, 1]$ with $\alpha_j^{+} < \alpha_j^{-}$ if $0 \leq \lambda_j < \frac{1}{4}$ and $\mathrm{Re}\,(\alpha_j^{\pm}) = \frac{1}{2}$ otherwise. It is shown that the series $\zeta_{\Delta_\Gamma}(w, z)$ converges absolutely for $\mathrm{Re}\,(w) > 1$, admits a meromorphic continuation to the whole plane $\mathbb{C}$ and is in particular holomorphic at $w = 0$ (see, e.g., [9–11]). Moreover, the determinant $\det\,(\Delta_\Gamma - s(1-s)) := \mathrm{Det}_1(\Delta_\Gamma - s(1-s))$ can be calculated as

$$\det\big(\Delta_\Gamma - s(1-s)\big) = G_\Gamma(s) Z_\Gamma(s) = \phi(s)^{g-1} Z_\Gamma(s). \tag{1.1}$$

Here, $\phi(s)$ is a meromorphic function defined by

$$\begin{aligned}\phi(s) &:= e^{-2\left(s-\frac{1}{2}\right)^2 - 4\left(s-\frac{1}{2}\right)\zeta'(0) + 4\zeta'(-1)} \Gamma(s)^{-2} G(s)^{-4} \\ &= e^{-2\left(s-\frac{1}{2}\right)^2} \Gamma_1(s)^{-2} \Gamma_2(s)^4\end{aligned}$$

with $\zeta(s)$, $\Gamma(s)$, $G(s)$ and $\Gamma_n(s)$ being the Riemann zeta function, the classical gamma function, the Barnes G-function (= a double gamma function [12]) and the Barnes multiple gamma function [13], respectively. The other factor $Z_\Gamma(s)$ in (1.1) is the Selberg zeta function defined by the Euler product

$$Z_\Gamma(s) := \prod_{P \in \mathrm{Prim}\,(\Gamma)} \prod_{n=0}^{\infty} \big(1 - N(P)^{-s-n}\big) \quad (\mathrm{Re}\,(s) > 1).$$

Here, $\mathrm{Prim}\,(\Gamma)$ is the set of all primitive hyperbolic conjugacy classes of Γ and $N(P)$ is the square of the larger eigenvalue of $P \in \mathrm{Prim}\,(\Gamma)$. It is known that $Z_\Gamma(s)$ can be continued analytically to the whole plane $\mathbb{C}$ and has the functional equation

$$Z_\Gamma(1-s) = \Big(S_1(s)^2 S_2(s)^{-4}\Big)^{g-1} Z_\Gamma(s), \tag{1.2}$$

where $S_n(s)$ is the normalized multiple sine function defined by (3.31) [14]. Notice that if we define the complete Selberg zeta function $\Xi_\Gamma(s)$ by

$$\Xi_\Gamma(s) := \big(\Gamma_2(s)^2 \Gamma_2(s+1)^2\big)^{g-1} Z_\Gamma(s),$$

then, the functional equation (1.2) is equivalent to

$$\Xi_\Gamma(1-s) = \Xi_\Gamma(s). \tag{1.3}$$

Moreover, it is known that $Z_\Gamma(s)$ has zeros at $s = 1, 0, -k$ for $k \in \mathbb{N}$ with multiplicity $1, 2g-1, 2(g-1)(2k+1)$, respectively (latest are called the trivial zeros because they also come from the gamma factor for the Riemann zeta function) and at $s = \alpha_j^{\pm}$ for $j \in \mathbb{N}$ (these are called the non-trivial zeros). In particular, $Z_\Gamma(s)$ satisfies an analogue of the Riemann hypothesis, i.e., all imaginary zeros of $Z_\Gamma(s)$ are located on the line $\mathrm{Re}\,(s) = \frac{1}{2}$. See [15] for arithmetic trial of a determinant expression for the Riemann zeta function.