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On Penrose integral formula and series expansion of k-regular functions on the quaternionic space $\mathbb{H}^n$ ☆

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ABSTRACT

The k-Cauchy–Fueter operator can be viewed as the restriction to the quaternionic space $\mathbb{H}^n$ of the holomorphic k-Cauchy–Fueter operator on $\mathbb{C}^{4n}$. A generalized Penrose integral formula gives the solutions to the holomorphic k-Cauchy–Fueter equations, and conversely, any holomorphic solution to these equations is given by this integral formula. By restriction to the quaternionic space $\mathbb{H}^n \subseteq \mathbb{C}^{4n}$, we find all k-regular functions. The integral formula also gives the series expansion of a k-regular function by homogeneous k-regular polynomials. In particular, the result holds for left regular functions, which are exactly 1-regular. It is almost elementary to show the k-regularity of the function given by the integral formula or such series, but the proof of the inverse part that any k-regular function can be provided by the integral formula or such series involves some tools of sheaf theory.

1. Introduction

The k-Cauchy–Fueter operators play important roles in quaternionic analysis (cf. [1–9] and references therein). Theorems in [7] proved by the second author imply that there was a 1–1 correspondence between the first cohomology of the sheaf $\mathcal{O}(-k-2)$ over an open subset of the projective space and the solutions to the k-Cauchy–Fueter equations on the quaternionic space $\mathbb{H}^n$. It is quite interesting to find the explicit integral formula realizing this correspondence, since it will allow us to find all solutions to k-Cauchy–Fueter equations, i.e., all k-regular functions.

We write a vector in $\mathbb{H}^n$ as $\mathbf{q} = (\mathbf{q}_0, \ldots, \mathbf{q}_{n-1})$ with $\mathbf{q}_l = x_{4l} + x_{4l+1}\mathbf{i} + x_{4l+2}\mathbf{j} + x_{4l+3}\mathbf{k} \in \mathbb{H}$, $l = 0, 1, \ldots, n-1$. The usual *Cauchy–Fueter operator* $\mathscr{D} : C^1(\mathbb{H}^n, \mathbb{H}) \to C(\mathbb{H}^n, \mathbb{H}^n)$ is defined as

$$\mathscr{D}f = \begin{pmatrix} \overline{\partial}_{\mathbf{q}_0} f \\ \vdots \\ \overline{\partial}_{\mathbf{q}_{n-1}} f \end{pmatrix}, \tag{1.1}$$

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for $f \in C^1(\mathbb{H}^n, \mathbb{H})$, where

$$\overline{\partial}_{\mathbf{q}_l} = \partial_{x_{4l}} + \mathbf{i}\partial_{x_{4l+1}} + \mathbf{j}\partial_{x_{4l+2}} + \mathbf{k}\partial_{x_{4l+3}}, \tag{1.2}$$

$l = 0, 1, \ldots, n-1$. A function $f : \mathbb{H}^n \to \mathbb{H}$ is called *(left) regular* if $\mathscr{D}f \equiv 0$ on $\mathbb{H}^n$. We want to find all left regular functions on $\mathbb{H}^n$.

1.1. The holomorphic k-Cauchy–Fueter operator

We begin with the definition of holomorphic k-Cauchy–Fueter operator on $\mathbb{C}^{4n}$ since the k-Cauchy–Fueter operator can be viewed as its restriction to the quaternionic space. We denote a point $\mathbf{z} \in \mathbb{C}^{4n}$ as $\mathbf{z} = (z^{AA'})$, and denote

$$\nabla_{AA'} := \partial_{z^{AA'}}, \tag{1.3}$$

the *holomorphic derivatives* on $\mathbb{C}^{4n}$, where $A = 0, 1, \ldots, 2n-1$, and $A' = 0', 1'$. An element of $\mathbb{C}^2$ is denoted by $(v_{0'}, v_{1'})$, while an element of the symmetric power $\odot^k \mathbb{C}^2$ is denoted by $(v_{A'_1\cdots A'_k})$ with $v_{A'_1\cdots A'_k} \in \mathbb{C}, A'_1, \ldots, A'_k = 0', 1'$, where $v_{A'_1\cdots A'_k}$ is invariant under the permutations of subscripts. Therefore,

$$\odot^k \mathbb{C}^2 \cong \mathbb{C}^{k+1},$$

and an element v of $\odot^k \mathbb{C}^2$ can be written as $v = (v_{0'\cdots 0'}, v_{1'0'\cdots 0'}, \ldots, v_{1'\cdots 1'})^t$, where ^{t} is the transpose.

For $k = 1, 2, \ldots$, define the *holomorphic k-Cauchy–Fueter operator*

$$\begin{array}{rccc} D^{(k)} : & \mathscr{H}(\mathbb{C}^{4n}, \mathbb{C}^{k+1}) & \longrightarrow & \mathscr{H}(\mathbb{C}^{4n}, \mathbb{C}^{2n} \otimes \mathbb{C}^k), \\ & \phi & \longmapsto & D^{(k)}\phi, \end{array}$$

where for $\phi = (\phi_{A'_1\cdots A'_k})$,

$$\left(D^{(k)}\phi\right)_{AA'_2\cdots A'_k} := \nabla_A^{A'_1}\phi_{A'_1A'_2\cdots A'_k}, \tag{1.4}$$

$A'_1, \ldots, A'_k = 0', 1', A = 0, 1, \ldots, 2n-1$, and for some complex vector space V, $\mathscr{H}(\mathbb{C}^{4n}, V)$ is the space of all V-valued holomorphic functions on $\mathbb{C}^{4n}$. Here and in the following we use Einstein convention of taking summation over repeated indices. The matrix

$$\epsilon = (\epsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.5}$$

is used to raise or lower indices, e.g., $\nabla_A^{A'_1}\epsilon_{A'_1A'_2} = \nabla_{AA'_2}$. In particular,

$$\nabla_A^{0'} = \nabla_{A1'}, \qquad \nabla_A^{1'} = -\nabla_{A0'}. \tag{1.6}$$

Then the *holomorphic-k*-Cauchy–Fueter equations $D^{(k)}\phi = 0$ can be written as

$$\nabla_{A1'}\phi_{0'A'_2\cdots A'_k} - \nabla_{A0'}\phi_{1'A'_2\cdots A'_k} = 0,$$

$A = 0, 1, \ldots, 2n-1, A'_2, \ldots, A'_k = 0', 1'$. In particular, the *holomorphic 1-Cauchy–Fueter equations* $D^{(1)}\phi = 0$ can be written as

$$\begin{aligned} D^{(1)}\begin{pmatrix} \phi_{0'} \\ \phi_{1'} \end{pmatrix} &= \begin{pmatrix} \nabla_{01'} & -\nabla_{00'} \\ \vdots & \vdots \\ \nabla_{(2n-1)1'} & -\nabla_{(2n-1)0'} \end{pmatrix}\begin{pmatrix} \phi_{0'} \\ \phi_{1'} \end{pmatrix} \\ &= \begin{pmatrix} \partial_{z^{01'}} & -\partial_{z^{00'}} \\ \vdots & \vdots \\ \partial_{z^{(2n-1)1'}} & -\partial_{z^{(2n-1)0'}} \end{pmatrix}\begin{pmatrix} \phi_{0'} \\ \phi_{1'} \end{pmatrix} = 0, \end{aligned}$$

where $\phi = (\phi_{0'}, \phi_{1'})^t \in \mathscr{H}(\mathbb{C}^{4n}, \mathbb{C}^2)$.