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Initial-boundary value problems for integrable evolution equations with 3×3 Lax pairs

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ABSTRACT

We present an approach for analyzing initial-boundary value problems for integrable equations whose Lax pairs involve 3×3 matrices. Whereas initial value problems for integrable equations can be analyzed by means of the classical Inverse Scattering Transform (IST), the presence of a boundary presents new challenges. Over the last fifteen years, an extension of the IST formalism developed by Fokas and his collaborators has been successful in analyzing boundary value problems for several of the most important integrable equations with 2×2 Lax pairs, such as the Korteweg–de Vries, the nonlinear Schrödinger, and the sine-Gordon equations. In this paper, we extend these ideas to the case of equations with Lax pairs involving 3×3 matrices.

1. Introduction

Several of the most important PDEs in mathematics and physics are integrable. Integrable PDEs can be analyzed by means of the Inverse Scattering Transform (IST) formalism, whose introduction was one of the most important developments in the theory of nonlinear PDEs in the 20th century. Until the 1990s the IST methodology was pursued almost entirely for pure initial value problems. However, in many laboratory and field situations, the wave motion is initiated by what corresponds to the imposition of boundary conditions rather than initial conditions. This naturally leads to the formulation of an initial-boundary value (IBV) problem instead of a pure initial value problem.

In 1997, Fokas announced a new unified approach for the analysis of IBV problems for linear and nonlinear integrable PDEs [1,2] (see also [3]). The Fokas method provides a generalization of the IST formalism from initial value to IBV problems, and over the last fifteen years, this method has been used to analyze boundary value problems for several of the most important integrable equations with 2×2 Lax pairs, such as the Korteweg–de Vries, the nonlinear Schrödinger, the sine-Gordon, and the stationary axisymmetric Einstein equations, see e.g. [2–12]. Just like the IST on the line, the unified method yields an expression for the solution $u(x, t)$ of an IBV problem in terms of the solution of a Riemann–Hilbert (RH) problem. In particular, the asymptotic behavior of $u(x, t)$ can be analyzed in an effective way by using this RH problem and by employing the nonlinear version of the steepest descent method introduced by Deift and Zhou [13].

The purpose of this paper is to develop a methodology for analyzing IBV problems for integrable evolution equations with Lax pairs involving 3×3 matrices. Although the transition from 2×2 to 3×3 matrix Lax pairs involves a number of novelties, the two main steps of the method of [1,2] remain the same: (a) Construct an integral representation of the solution characterized via a matrix RH problem formulated on the Riemann k-sphere, where k denotes the spectral parameter of the Lax pair. This representation involves, in general, some unknown boundary values, thus the solution formula is not yet effective. (b) Characterize the unknown boundary values by analyzing the so-called global relation. In general, the characterization of the unknown boundary values involves the solution of a nonlinear problem. However, for so-called *linearizable* boundary conditions, this problem can be by-passed since the unknown boundary values can be eliminated using only algebraic manipulations.

In this paper, we will show how steps (a) and (b) can be implemented for a prototypical example of an equation with a 3×3 Lax pair. We expect that a similar analysis will apply also to other integrable equations with 3×3 Lax pairs, both on the half-line and on the interval, although in certain cases additional technical difficulties will arise. For example, for the Boussinesq equation the behavior of the eigenfunctions near the point $k = 0$

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requires special attention.[1] Physically relevant equations with 3×3 Lax pairs include the Boussinesq, Degasperis–Procesi [15], Kaup–Kupershmidt [16], Sasa–Satsuma [17], Sawada–Kotera [18], two-component vector nonlinear Schrödinger [19], and 3-wave resonant interaction [20] equations.

1.1. The transition from 2×2 to 3×3 Lax pairs

Let us comment on some of the implications of the transition from 2×2 to 3×3 Lax pairs. The implementation of the step (a) mentioned earlier in the case of 2×2 matrix Lax pairs is achieved by introducing eigenfunctions $\{\mu_j(x, t, k)\}$ which are defined by integration from the 'corners' of the physical domain, see [3]. The column vectors of the μ_j's are bounded and analytic in different sectors of the complex k-plane and these column vectors are easily combined into a sectionally analytic function suitable for the formulation of a RH problem. However, for a Lax pair involving 3×3-matrices, the bounded and analytic eigenfunctions suitable for the formulation of a RH problem involve rather complicated combinations of the entries of the μ_j's. Moreover, because of the limited domains of boundedness of the μ_j's, the boundedness properties of these combinations are not evident. We will therefore use a different approach for finding these combinations and their boundedness domains. Instead of taking the μ_j's as our starting point, we will define analytic eigenfunctions, denoted by $\{M_n(x, t, k)\}$, via integral equations which involve integration from all three corners simultaneously. In the absence of bound states, this formulation is adequate. However, since the integral equations defining the eigenfunctions now are of Fredholm rather than Volterra type, there may exist points $\{k_j\}$, $k_j \in \mathbb{C}$, at which the M_n's have singularities. In order to deal with these singularities (which are related to the existence of solitons), we will relate the M_n's to the μ_j's by solving a matrix factorization problem.

Another difference in the implementation of step (a) is that the RH problem in the case of a 2×2 Lax pair splits the complex k-plane into 4 sectors, whereas a larger number of sectors is in general required in the case of 3×3 Lax pairs, e.g. for our main example, the RH problem splits the complex k-plane into 12 sectors.

The implementation of the step (b) mentioned earlier involves eliminating the unknown boundary values from the formulation of the RH problem. In the case of linearizable boundary conditions for equations with 2×2 matrices, this elimination is achieved by algebraic manipulation of the so-called global relation and the equations obtained from the global relation under certain transformations in the k-plane. As shown in Section 4, similar ideas can be used to analyze linearizable boundary conditions in the case of 3×3 Lax pairs. However, in the 3×3 case the algebraic relations cannot always be directly used to eliminate the unknown boundary values from the definition of the jump matrices. Instead, we will first analytically extend the domain of definition of the jump matrix, before we utilize the algebraic relations. Finally, we perform another analytic continuation to find the expression for the jump matrix on the relevant contour. The analyticity domains of the involved matrices are just sufficient to allow for this approach.

Let us finally point out that some pioneering works on the theory of inverse scattering for initial-value problems for equations with 3×3 Lax pairs are [16,21,22]. Other examples of the use of piecewise analytic solutions in studying the inverse problems for $n \times n$ systems (particularly for $n = 4$) can be found in [23–25].

[1] A similar situation arises for KdV for which the eigenfunctions have poles at $k = 0$, see [14].

1.2. Organization of the paper

Our main example is introduced in Section 1.3. The spectral analysis of the associated Lax pair is performed in Section 2. In Section 3, we formulate the main RH problem, and this concludes the implementation of step (a) above.

Step (b) is implemented in Sections 4 and 5. In Section 4, we consider linearizable boundary conditions, whereas nonlinearizable boundary conditions are analyzed in Section 5. Section 6 contains some concluding remarks. In Appendix A, we explain the relationship between the constructions of this paper and the formalism of [1,2] for 2×2 Lax pairs. In Appendix B, we use an extension of the standard Fredholm theory to study a set of integral equations.

1.3. The main example

To be concrete, we will consider the system

$$\begin{cases} iq_t + \frac{1}{\sqrt{3}} q_{xx} + 2irr_x = 0, \\ ir_t - \frac{1}{\sqrt{3}} r_{xx} + 2iqq_x = 0, \end{cases} \tag{1.1}$$

where $q(x, t)$ and $r(x, t)$ are complex-valued functions of $(x, t) \in \Omega$, with Ω denoting the half-line domain

$$\Omega = \{0 < x < \infty,\ 0 < t < T\}$$

and $T > 0$ being a fixed final time. This system is the compatibility condition of the 3×3 Lax pair

$$\begin{cases} \psi_x - kJ\psi = V_1\psi, \\ \psi_t + k^2J^2\psi = V_2\psi, \end{cases} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad \omega = e^{\frac{2\pi i}{3}}, \tag{1.2}$$

where $\psi(x, t, k)$ is a 3×3-matrix valued function, $k \in \mathbb{C}$ is the spectral parameter, and $\{V_1(x, t), V_2(x, t, k)\}$ are 3×3-matrix valued functions given by

$$V_1 = \begin{pmatrix} 0 & q & r \\ r & 0 & q \\ q & r & 0 \end{pmatrix}, \qquad V_2(x, t, k) = kV_2^{(1)}(x, t) + V_2^{(0)}(x, t), \tag{1.3}$$

$$V_2^{(1)} = \begin{pmatrix} 0 & \omega q & \omega^2 r \\ \omega r & 0 & q \\ \omega^2 q & r & 0 \end{pmatrix},$$

$$V_2^{(0)} = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & q_x & -r_x \\ -r_x & 0 & q_x \\ q_x & -r_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & r^2 & q^2 \\ q^2 & 0 & r^2 \\ r^2 & q^2 & 0 \end{pmatrix}.$$

We will denote the initial data of (1.1) by $\{q_0(x), r_0(x)\}$, while the Dirichlet and Neumann boundary values will be denoted by $\{g_0(t), h_0(t)\}$ and $\{g_1(t), h_1(t)\}$, respectively, i.e.

$$\begin{aligned} &q(x, 0) = q_0(x), \quad r(x, 0) = r_0(x), \quad 0 < x < \infty; \\ &q(0, t) = g_0(t), \quad r(0, t) = h_0(t), \\ &q_x(0, t) = g_1(t), \quad r_x(0, t) = h_1(t), \quad 0 < t < T. \end{aligned} \tag{1.4}$$

Remark 1.1. 1. We have chosen (1.1) as our main example, because the Lax pair (1.2) is a natural 3×3 generalization of the 2×2 Lax pair for the nonlinear Schrödinger equation. Indeed, the x and t parts of the Lax pair for NLS involve the matrices $k\sigma_3$ and $k^2\sigma_3$, respectively, where σ_3 is the diagonal matrix whose entries are the second roots of unity, i.e. $\sigma_3 = \text{diag}(1, -1)$. Analogously, the x and t parts of (1.2) involve the matrices kJ and k^2J^2, where J is the diagonal matrix whose entries are the third roots of unity.