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Painlevé IV: A numerical study of the fundamental domain and beyond

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HIGHLIGHTS

- First comprehensive study of the solution space to the fourth Painlevé equation.
- Extensions to parameter regimes that were unreachable by previous approaches.
- Previously known solutions found to be 'non-typical' of the general case.
- Solutions featuring singularity-free half-planes explored.

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ABSTRACT

The six Painlevé equations were introduced over a century ago, motivated by rather theoretical considerations. Over the last several decades, these equations and their solutions, known as the Painlevé transcendents, have been found to play an increasingly central role in numerous areas of mathematical physics. Due to extensive dense pole fields in the complex plane, their numerical evaluation remained challenging until the recent introduction of a fast 'pole field solver' (Fornberg and Weideman, 2011). The fourth Painlevé equation has two free parameters in its coefficients, as well as two free initial conditions. After summarizing key analytical results for P_{IV}, the present study applies this new computational tool to the fundamental domain and a surrounding region of the parameter space. We confirm existing analytic and asymptotic knowledge about the equation and also explore solution regimes which have not been described in the previous literature.

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1. Introduction

With the increasing presence of the Painlevé equations in the reduction of partial differential equations (PDEs) [1–4] and the subjects of combinatorics [5–7], orthogonal polynomials [8–12], statistical physics [13–18], integrable continuous dynamical systems [19,20] and quantum physics [21–24] a greater understanding of the solution space for each of the six equations is important. A collection of applications specific to the P_{IV} equation is presented in [25]. In the past, solutions that are pole free along the real axis have proven to be particularly relevant. As a resource for the future, one present goal has been to identify such cases, as well as those with pole-free sectors in the complex-plane, throughout the P_{IV} equations four-parameter solution space.

The solutions of the six Painlevé equations (P_I–P_{VI}) are free from movable branch points, but with the possibility of movable poles or movable isolated essential singularities ([26], Section 32.2). This Painlevé property inspired the introduction of a novel numerical approach [27] – combining a Padé-based ODE solver [28] with a partly randomized integration path strategy – that allowed for the first time rapid numerical solutions of the Painlevé equations over extended regions in the complex plane. It was first used for P_I [27] and later for P_{II} [29]. It was then applied to the fourth Painlevé equation

$$\frac{d^2}{dz^2}u(z) = \frac{1}{2u(z)}\left(\frac{d}{dz}u(z)\right)^2 + \frac{3}{2}u(z)^3 + 4zu(z)^2 + 2\left(z^2-\alpha\right)u(z) + \frac{\beta}{u(z)}, \quad (1)$$

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in the special case of $\alpha = \beta = 0$ [30]. As in these three previous numerical studies, computational explorations in this paper are limited to solutions $u(z)$ that are real when z is real, although some of the presented theory includes solutions that are not always real on the real axis.

For a small portion of the two-dimensional (α, β)-parameter space there exist examples of solutions expressible as rational functions or in terms of special functions, such as the parabolic cylinder function. These well-documented solutions appear, however, as only isolated points or one-parameter families in the four-dimensional space of parameters and initial conditions (ICs). Much of the present study is focused on the distribution of singularities for solutions to (1). These are all first-order poles with residue $+1$ or -1.

The solutions presented in this paper are parameterized by α, β and the values for $u(0)$ and $u'(0)$. Another way to parameterize the solution space is through four "Stokes multipliers", which provides a link between the P_{IV} equation and a certain Riemann–Hilbert problem [31–35]. This approach is particularly well suited for analytical work such as connection formulas and far-field asymptotics. Distant pole field structures can also be approximated via suitable transformations [36,37] (however, the present focus is more on pole-free regions). The two parameterization approaches can be related to each other utilizing, for instance, the software RHPackage [38] to solve the Riemann–Hilbert problem (given a set of Stokes multipliers and parameters to define α and β) to determine the corresponding set of values for $u(0)$ and $u'(0)$.

1.1. Organization of the paper

Section 2 recalls some available theory, including symmetries in P_{IV} and different solution transformations. Section 3 discusses closed form solutions of P_{IV}, in particular solutions in terms of rational and elementary special functions and also the asymptotic behaviors presented in the literature. This is followed in Section 4 by the numerical approach used here to explore the much larger space of solutions for which no closed form solutions are available. Sections 5 and 6 describe such explorations of the parameter and solution spaces, first highlighting the "fundamental domain" and then extending into inspections of the previously unexplored region of $\beta > 0$, for which no instances of closed form solutions or transformations have been reported.

2. Symmetries and solution hierarchies

This section describes the known symmetries in the P_{IV} equation and transformations that relate solutions for different parameter choices.

2.1. Symmetries in the equation

Let $P_{IV}(\alpha, \beta)$ be the set of all solutions of (1) for the particular α and β. Direct inspection of (1) shows that if $u(z) \in P_{IV}(\alpha, \beta)$, then [39]

$$-u(-z) \in P_{IV}(\alpha, \beta), \tag{2}$$

$$-iu(-iz) \in P_{IV}(-\alpha, \beta), \text{ and} \tag{3}$$

$$iu(iz) \in P_{IV}(-\alpha, \beta). \tag{4}$$

Incidentally the first of these symmetries also holds for P_{III} (for all parameter choices), but never for any of the other Painlevé equations. Due to these symmetries, any solution presented in this paper has at least one other counterpart for the same choice of α and β.

2.2. The Bäcklund and Schlesinger transformations

The equations P_{II} through P_{VI} have collections of transformations relating solutions for given parameters to those of different choices. For instance, [39–42] collectively present sixteen such transformations for P_{IV}. Some of these transformations were not always presented correctly. Updated expressions along with computational verification of their forms can be found in [43].

3. Closed form solutions and asymptotic approximations

Before discussing the closed form solutions presented in the literature, we note again that these at most form two-dimensional manifolds in the four-dimensional solution space. That is, they provide a very limited view of the solution types that are possible.

3.1. Rational solutions

The P_{IV} equation has six different sequences of parameter choices leading to rational solutions expressible in terms of either Generalized Hermite or Generalized Okamoto polynomials [44], with two particular choices leading to the only known entire solutions, $-2z$ and $-(2/3)z$. The locations of the parameter choices in the (α, β)-plane leading to such solutions are shown in Fig. 1 as dark (blue) and light (yellow) hexagrams for Generalized Hermite and Generalized Okamoto polynomials, respectively.

3.2. Special function solutions

In addition to the rational solutions, P_{IV} admits solutions that are described by combinations of special functions; cf. [26], chapters 12 and 13. This includes solutions expressible in terms of parabolic cylinder functions, $D_\nu(\zeta)$ [5,45], and, as discovered more recently, solutions in terms of the confluent hypergeometric function, ${}_1F_1(a, b; \zeta)$, [23,24]. In either case, for each of the appropriate choices of α and β there is a one parameter family of solutions that are expressible in terms of these special functions. Fig. 1 displays the locations of all such parameter choices as black curves.

Three distinct types of solutions have been proposed in the form of determinants involving parabolic cylinder functions [5,45]. However, only one of these expressions has been confirmed numerically [43].

3.3. Asymptotic approximations

Beyond the known closed form solutions, it is noted in [26], section 32.11, that when $\beta = 0$, there are solutions that decay asymptotically along the real axis either as $z \to +\infty$ or $z \to -\infty$. These solutions result from assuming that the second derivative term in (1) is negligible.

When assuming instead that both the first and second derivative terms in (1) are negligible, the method of dominant balance (see, e.g., [46], section 3.4) leads to the quartic equation

$$\frac{3}{2}\hat{w}(z)^4 + 4z\hat{w}(z)^3 + 2(z^2 - \alpha)\hat{w}(z)^2 + \beta = 0. \tag{5}$$

Each root of (5) provides a leading asymptotic term for solutions that are smooth as $z \to \pm\infty$. Any number of further terms then follow by substitution into (1). For instance, (6) through (9) illustrate the first two terms.

$$w^+_{+1}(z; \alpha, \beta) = \frac{\sqrt{-2\beta}}{2z} + \frac{\alpha\sqrt{-2\beta} + 2\beta}{4z^3} + O\left(\frac{1}{z^5}\right) \tag{6}$$

$$w^+_{-1}(z; \alpha, \beta) = -\frac{\sqrt{-2\beta}}{2z} + \frac{-\alpha\sqrt{-2\beta} + 2\beta}{4z^3} + O\left(\frac{1}{z^5}\right) \tag{7}$$

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