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Periodic orbits from second order perturbation via rational trigonometric integrals

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H I G H L I G H T S

- Along the number, we study the shape and the period of the perturbed periodic orbits.
- First a second order studies are given for concrete planar vector fields.
- Same technique apply simultaneously to Abel equations and planar vector fields.
- The computations share the same integrals of rational trigonometric functions.
- Isochronous quadratic systems have at most 2 limit cycles up to second order.

A R T I C L E   I N F O

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The second order Poincaré-Pontryagin-Melnikov perturbation theory is used in this paper to study the number of bifurcated periodic orbits from certain centers. This approach also allows us to give the shape and the period up to the first order. We address these problems for some classes of Abel differential equations and quadratic isochronous vector fields in the plane. We prove that two is the maximum number of hyperbolic periodic orbits bifurcating from the isochronous quadratic centers with a birational linearization under quadratic perturbations of second order. In particular the configurations (2, 0) and (1, 1) are realizable when two centers are perturbed simultaneously. The required computations show that all the considered families share the same iterated rational trigonometric integrals.

1. Introduction

In this paper we study the number, the shape and the period of closed trajectories bifurcating from the period annuli for some families of differential equations. We focus our attention on the second order analysis of the perturbed equation

$$\frac{dr}{d\theta} = a_0(r,\theta) + \varepsilon\, a_1(r,\theta) + \varepsilon^2\, a_2(r,\theta) + O(\varepsilon^3). \qquad (1)$$

Here a_i is an analytic function, 2π-periodic in $\theta \in [-\pi, \pi]$, for $i = 0, 1, 2$. We denote by $r(\theta; \rho, \varepsilon)$ the solution of (1) satisfying $r(-\pi; \rho, 0) = \rho$. We will assume throughout this paper that ε is small enough. The power series in ε of this solution is written as

$$r(\theta; \rho, \varepsilon) = r_0(\theta; \rho) + \varepsilon\, r_1(\theta; \rho) + \varepsilon^2\, r_2(\theta; \rho) + O(\varepsilon^3). \qquad (2)$$

We refer to this expansion as the *shape of the orbit*. Additionally we assume that, when $\varepsilon = 0$, Eq. (1) has a *period annulus*; that is, an open continuum neighborhood of periodic solutions. In particular, there exists an open interval where the function $r_0(\theta; \rho)$ is 2π-periodic in $\theta \in [-\pi, \pi]$, for every ρ in this interval.

In the concrete case $a_0(r, \theta) = a(\theta)r^2$ we have $r_0(\theta; \rho) = \rho/(1 - \rho A(\theta))$ where $A(\theta) = \int_{-\pi}^{\theta} a(\psi)\, d\psi$. Note that A is also an analytic 2π-periodic function. Furthermore, in a first order analysis of the first return map associated to the period annulus, the

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Poincaré–Pontryagin–Melnikov function takes the form

$$F_1(\rho) = \int_{-\pi}^{\pi} a_1\left(\frac{\rho}{1-\rho A(\theta)}, \theta\right)(1-\rho A(\theta))^2\, d\theta = \sum_{i=0}^{\infty}\int_{-\pi}^{\pi} \frac{a_{1,i}(\theta)\rho^i}{(1-\rho A(\theta))^{i-2}}\, d\theta, \quad (3)$$

when $a_1(r,\theta) = \sum_{i=0}^{\infty} a_{1,i}(\theta) r^i$. It is well known that each simple zero of F_1 gives rise to a 2π-periodic solution bifurcating from $r_0(\theta;\rho)$; see [1].

Observe that F_1 is defined through an Abelian integral and, in general, the computation of this type of integrals is quite difficult because it involves transcendental functions. As it can be seen in (3), to obtain F_1 we integrate rational trigonometric functions and hence some non-rational terms appear in the calculation but, since we integrate over the usual circles and by using the formulas of Appendix, we have their expressions.

Also it is known that if this integral is vanishing identically then, in the second order analysis, the Poincaré–Pontryagin–Melnikov function is no longer an Abelian integral, see [2–5], and moreover its study involves the computation of the so-called iterated integrals in the sense of Chen–Gavrilov, see [2]. Concerning the infinitesimal Hilbert's 16th problem this case is a more interesting one since the Poincaré–Pontryagin–Melnikov function can have more zeros, i.e. we can produce (in general) more limit cycles than at the ones at first order; see [6–8]. More details about higher order Poincaré–Pontryagin–Melnikov theory can be found in [9].

In this paper we focus on getting the second order Poincaré–Pontryagin–Melnikov perturbation of some families of centers. The iterated integrals of rational functions that are necessary to reach the second order cannot always be expressed in terms of elementary functions. When this happens we say that the integrals are non-explicit. The calculation of these integrals involves extra difficulties compared with the calculations at the first order analysis. Hence, in general to get a higher order study may not be as easy. More concretely, since we have considered rational perturbations up to second order in ε, the computation of the Poincaré–Pontryagin–Melnikov functions involves the integrals

$$\int_{-\pi}^{\theta} \frac{\sin(k\psi)\,\varphi(r,\psi)}{(1+r\cos\psi)^{\ell}}\, d\psi, \qquad \int_{-\pi}^{\theta} \frac{\cos(k\psi)\,\varphi(r,\psi)}{(1+r\cos\psi)^{\ell}}\, d\psi, \quad (4)$$

where φ represents either the constant 1 or the λ or ϕ function, or even their corresponding primitives, being

$$\phi(r,\theta) = \frac{1}{\sqrt{1-r^2}}\left(\theta - 2\arctan\left(\sqrt{\frac{1-r}{1+r}}\tan\left(\frac{\theta}{2}\right)\right)\right), \quad (5)$$
$$\lambda(r,\theta) = \log(1+r\cos\theta),$$

where $r \in (-1,1)$ and $\theta \in [-\pi,\pi]$. From the functions defined in (5) we say that the iterated integrals that appear in this paper are explicit since they can be expressed using an extension of the set of elementary functions with the functions ϕ and λ.

The goals of this paper are to get the shape, number and period of the hyperbolic periodic solutions bifurcating from the period annulus for some families of Eqs. (1). We use a second order approach based on a generalization of the method introduced in [10] and improved in [11]. In the former reference the first non-zero derivative of the return map associated with orbits of a perturbed Hamiltonian system, $dH + \varepsilon\,\omega = 0$ where $H(x,y) = x^2 + y^2$, is done. In the latter one, the method is extended to radial Hamiltonians to study the shape of an orbit as well. A higher order study when ω is a polynomial one form is carried out in [12] because all the iterated primitives are obtained in terms of elementary functions.

The averaging theory can be an equivalent approach to obtain bifurcated periodic solutions for equations of type (1). Recent works, where higher order analysis of this theory is developed and applied to study the periodic orbits that some polynomial vector fields present, are [13,14].

In Section 2 we study the shape, number and period of the isolated periodic solutions concerning a subclass of Abel differential equations. In Section 3, as a second application, we examine some families of planar polynomial differential equations. In Section 4 simultaneous bifurcation of limit cycles, from two continua of periodic orbits in quadratic isochronous centers, is considered. In Section 5 we summarize what the main impediments are to get higher order. Finally, in the Appendix we write down the expressions and properties of the rational trigonometric integrals shared by the families in this paper, the use of which allows us to achieve the higher order studies. As we will see in the proof of the results, the integrals of Appendix are useful for all the computations because the involved denominators are powers of polynomials of degree one in $\cos\theta$. Below we detail the contents of Sections 2–4.

In Section 2 we consider the differential equation

$$\frac{dr}{d\theta} = a(\theta)\, r^2 + \varepsilon \sum_{i=0}^{m} P_i(\theta) r^i + \varepsilon^2 \sum_{i=0}^{m} Q_i(\theta) r^i, \quad (6)$$

where $a(\theta)$, $P_i(\theta)$ and $Q_i(\theta)$ are analytic and 2π-periodic in $\theta \in [-\pi,\pi]$. We define the functions

$$\begin{aligned} F_1(r) &= \sum_{i=0}^{m}\int_{-\pi}^{\pi} r^i P_i(\theta)(1-rA(\theta))^{2-i}\, d\theta,\\ F_2(r) &= \sum_{i=0}^{m}\int_{-\pi}^{\pi} r^i Q_i(\theta)(1-rA(\theta))^{2-i}\, d\theta \\ &\quad + \sum_{i=0}^{m}\int_{-\pi}^{\pi} i r^{i-1} W_1(r,\theta) P_i(\theta)(1-rA(\theta))^{1-i}\, d\theta \\ &\quad + \int_{-\pi}^{\pi} a(\theta)(W_1(r,\theta))^2(1-rA(\theta))^{-2}\, d\theta, \end{aligned} \quad (7)$$

where $A(\theta) = \int_{-\pi}^{\theta} a(\psi)\, d\psi$ and $W_1(r,\theta) = \sum_{i=0}^{m}\int_{-\pi}^{\theta} r^i P_i(\psi)(1-rA(\psi))^{2-i}\, d\psi$.

From the above definitions we state next theorem which generalizes some of the results given by Françoise in [15].

Theorem 1. *Consider the differential equation (6) and the corresponding functions given in (7) with a 2π-periodic function $A(\theta)$, $\theta \in [-\pi,\pi]$.*

(i) *If $F_1(\rho) = 0$, and $F_1'(\rho) \neq 0$, then (6) has a hyperbolic 2π-periodic orbit which tends to $\rho(1-\rho A(\theta))^{-1}$ when ε goes to zero, and it is written as*

$$r(\theta;\rho,\varepsilon) = \frac{\rho}{1-\rho A(\theta)} + \varepsilon\, \frac{r_1(-\pi;\rho) + W_1(\rho,\theta)}{(1-\rho A(\theta))^2} + O(\varepsilon^2)$$

where $r_1(-\pi;\rho)$ is the solution of the equation $F_2(\rho) = 0$.

(ii) *When $F_1 = 0$, if $F_2(\rho) = 0$ and $F_2'(\rho) \neq 0$ then (6) has a hyperbolic 2π-periodic orbit that tends to $\rho(1-\rho A(\theta))^{-1}$ when ε goes to zero.*

We observe that, from the statement (i) of above theorem, shape up to the first order term in ε of a limit cycle needs a second order study to be determined completely. This can be seen in all the examples of next sections. Moreover, the functions F_1 and F_2 defined in (7) are the first and second Poincaré–Pontryagin–Melnikov functions for Eq. (6). As an application, next proposition bounds, up to the second order study, the number of periodic solutions of a polynomial perturbation of Eq. (6) for a class of Abel equations with $a(\theta) = \sin\theta$. The first order analysis for a restricted perturbation is shown in the previous works [16,17], when n is arbitrary.