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Dynamics of the wave turbulence spectrum in vibrating plates: A numerical investigation using a conservative finite difference scheme

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HIGHLIGHTS

- Non-stationary wave turbulence in a vibrating plate is numerically studied.
- Self-similar dynamics of the spectra are found with and without periodic external forcing.
- The self-similar solutions are in agreement with the kinetic equation.
- A realistic geometric imperfection is shown to have no effect on the global properties.

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ABSTRACT

The dynamics of the local kinetic energy spectrum of an elastic plate vibrating in a wave turbulence (WT) regime is investigated with a finite difference, energy-conserving scheme. The numerical method allows the simulation of pointwise forcing together with realistic boundary conditions, a set-up which is close to experimental conditions. In the absence of damping, the framework of non-stationary wave turbulence is used. Numerical simulations show the presence of a front propagating to high frequencies, leaving a steady spectrum in its wake. Self-similar dynamics of the spectra are found with and without periodic external forcing. For the periodic forcing, the mean injected power is found to be constant, and the frequency at the cascade front evolves linearly with time resulting in a increase of the total energy. For the free turbulence, the energy contained in the cascade remains constant while the frequency front increases as $t^{1/3}$. These self-similar solutions are found to be in accordance with the kinetic equation derived from the von Kármán plate equations. The effect of the pointwise forcing is observable and introduces a steeper slope at low frequencies, as compared to the unforced case. The presence of a realistic geometric imperfection of the plate is found to have no effect on the global properties of the spectra dynamics. The steeper slope brought by the external forcing is shown to be still observable in a more realistic case where damping is added.

1. Introduction

Wave Turbulence (WT) describes a system of waves interacting nonlinearly away from thermodynamical equilibrium [1,2]. Although the system under study is composed of waves only, the term "turbulence" is used here in analogy with hydrodynamic turbulence, where the energy of the system is transferred through scales (referred to as a cascade) resulting in a large bandwidth energy spectrum. A particular property is that, for WT systems, the form of the spectrum can be derived analytically [3] and not just in terms of dimensional analysis as for the Kolmogorov 41 theory of hydrodynamics turbulence [4]. Using the assumption of weak nonlinearity, and an appropriate separation of timescales, a natural closure arises leading to an analytical expression for the equation for the second order moment (*e.g.* the kinetic energy spectrum). Solutions to this equation lead to two physically different scenarios: the first one represents the system at equilibrium, where the total energy of the system is equally spread among all the Fourier components of the system (known as the modes), and thus corresponding to a Rayleigh–Jeans type of spectrum. The second scenario is out-of-equilibrium and leads to the Kolmogorov–Zakharov spectrum that describes a flux of energy from the injection scale, where energy is input in the system, to the dissipation scale such as in hydrodynamics turbulence. In the latter scenario the modes re-

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ceive and give energy to adjacent modes, thus creating a cascade of energy through scales. WT formalism has been applied to many systems in a variety of contexts, ranging from quantum-mechanical to astrophysical systems, and includes many systems encountered in the ordinary world. An exhaustive list may be found in [1]; here some examples are recalled: capillary [5,6] and surface gravity waves [7–9], Alfvén waves [10,11], and Kelvin waves [12,13].

Flexural waves produced by large amplitude vibrations of elastic plates have been studied within the framework of the wave turbulence theory [14] applied to the von Kármán equation [15,16] for the transverse displacement w. The analytical Kolmogorov–Zakharov spectrum is then given by

$$P_v(f) = \frac{Ch}{(1-\nu^2)^{2/3}}\varepsilon_c^{1/3}\log^{1/3}\left(\frac{f_c^\star}{f}\right); \tag{1}$$

where ε_c is the constant flux of energy transferred through scales, P_v refers to the power spectrum of the transverse velocity $v = \dot{w}$, h is the thickness of the plate, ν Poisson's ratio of the material, and C a constant. Because the theory is fully inertial, $f_c^\star$ is the frequency at which energy is removed from the system. In experiments, this is ensured by the damping of the plate. At first order the spectrum is flat, but with a log-correction in the inertial range of frequencies. The WT theoretical result has been compared to experiments [17,18], showing discrepancies regarding the shape and scaling of the spectrum with the energy flux. Thus, recent work has focused on the investigation of the possible causes for such discrepancies. Experimentally, the wave-structure and dispersion relation was checked in [18], leading to the conclusion that the nonlinear vibrations of a plate are indeed due to a set of waves following the theoretical (linear) dispersion relation. The correct separation of timescales, necessary assumption for the WT theory, was verified in [19]. A first discrepancy effect was observed in [20], showing that the local forcing of the shaker is responsible for a steeper slope in the supposed inertial range of the energy spectra. More recently, damping has also been shown to be the cause for a steeper slope of the spectrum, indicating that the inertial range might not exist for thin plates used in experiments, rendering then meaningless any comparison with the WT theory [21]. From the numerical standpoint, it is worth mentioning that all the numerical methods used so far are spectral schemes [14,22,21,23–25]. Hence the forcing is in the Fourier space, a feature that is different from a pointwise excitation used in experimental conditions. All available numerical results recover the KZ spectrum of Eq. (1) when the damping is localized at high frequency only. However, when realistic damping is added, see *e.g.* [21,23], the same conclusions as for the experiment are met.

Other sources of discrepancies have not been addressed yet, such as the finite size effects or the possibility of three wave interactions (quadratic nonlinearities) in real plates. Because of the $w \rightarrow -w$ symmetry of the von Kármán equation, these nonlinearities are not taken into account in [14]. Indeed, geometrical imperfections are unavoidable in real plates, and they are known to break this symmetry and to produce quadratic nonlinearities [26,27]. In particular, it has been shown in [28,29] that imperfections play an important role in the transition scenario to turbulence and favor instabilities and the appearance of quasiperiodic vibrations.

The numerical method used in this work relies on a finite difference, time domain, energy-conserving scheme [30,29]. The main advantages are that: (i) the time-stepping integration method conserves energy up to machine accuracy, so that essential properties of the underlying continuous Hamiltonian systems are preserved by the discretization [31]; (ii) the external forcing is pointwise in space just as in the real experiments; (iii) realistic boundary conditions can be implemented instead of using periodic boundary conditions as considered by previous numerical investigations using spectral methods [14,22,25].

The aim of this article is to investigate numerically wave turbulence produced by the von Kármán plate equations. With a numerical scheme close to experimental conditions, unavoidable effects in real experiments such as pointwise forcing and geometric imperfections can be accounted for. In order to properly distinguish the different effects, most of the presented results are obtained in the absence of damping, where the framework of non-stationary wave turbulence should be used [32,33]. The theory predicts self-similar dynamics of the spectra with a front propagating to higher frequencies. Such propagation has been observed for surface gravity waves in experiments [34]. On the contrary, capillary turbulence [35,36] exhibits a decay that begins from the high frequency end of the spectral range. The discrepancy with the self-similar theory of wave turbulence is ascribed to the presence of finite damping at all frequencies of the wave system [35,37].

The article is organized as follows: the governing equations together with the numerical approach are described in Section 2. Section 3 presents the data analysis tools used to study the spectral dynamics. The main results are given in Section 4. Periodically forced turbulence for a perfect plate is first considered. A self-similar propagation of a steep front towards the high frequencies, leaving in its wake a steady spectrum, is observed. The frequency of the front is found to evolve linearly with time. The presence of realistic geometric imperfections is then taken into account and shown to have no influence on the spectral dynamics. In Section 4.2, the case of a free, undamped turbulence is exhibited. In that case, self-similar dynamics of the spectra are also observed, but now the front evolves with time as $t^{1/3}$. Self-similar solutions derived from the kinetic equation are found to display the same dependences, thus validating the numerical results that give in addition the shape of the self-similar function. The pointwise forcing is found to influence the shape of the universal spectrum left in the wake of the front, with a steeper slope for the forced case. Finally, the effect of the pointwise forcing, underlined in the undamped cases, is confirmed in Section 4.3, where a decaying turbulence with a simple frequency-independent damping law is addressed. Discussion and concluding remarks appear in Section 5.

2. Dynamical equations

2.1. Continuous time and space equations

The system under study is a rectangular elastic plate of thickness h, dimensions L_x, L_y, volume density ρ, Poisson's ratio ν and Young's modulus E. Its flexural rigidity is defined as $D = \frac{Eh^3}{12(1-\nu^2)}$. The dynamics of weakly nonlinear waves for the transverse displacement $w(\mathbf{x}, t)$ can be described by the von Kármán equations [15,16]. The general case of an imperfect plate is here considered. If $w_0(\mathbf{x})$ denotes the initial (static) imperfection, then the equations of motion read [26,38,27]

$$D\Delta\Delta w + \rho h\ddot{w} = L(w + w_0, F) + \mathcal{F}(\mathbf{x}, t) - R(\dot{w}, t), \tag{2a}$$

$$\Delta\Delta F = -\frac{Eh}{2}L(w + 2w_0, w), \tag{2b}$$

where Δ is the Laplacian operator, $\Delta a(\mathbf{x}) = a_{,xx} + a_{,yy}$, and $L(\cdot, \cdot)$ is the bilinear symmetric von Kármán operator, $L(a(\mathbf{x}), b(\mathbf{x})) = a_{,xx}\, b_{,yy} + a_{,yy}\, b_{,xx} - 2a_{,xy}\, b_{,xy}$. $F(\mathbf{x}, t)$ is an auxiliary function called the Airy stress function which encapsulates the behavior of the plate in the in-plane direction, $R(\mathbf{x}, t)$ is a loss factor of some kind which will be specified shortly and $\mathcal{F}(\mathbf{x}, t)$ is the external excitation load. In this work, the material parameters are chosen to correspond to a steel plate; thus $E = 2 \times 10^{11}$ Pa, $\rho = 7860$ kg/m^3, $\nu = 0.3$. The other geometrical and physical parameters will be reported case by case.