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Extreme value laws for dynamical systems under observational noise

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HIGHLIGHTS

- We provide a full extreme value theory for dynamical systems perturbed with instrument-like-error.
- Numerical experiments support the theoretical findings.
- Fractal dimensions can be recovered in perturbed systems.
- The theory allows for studying recurrences on finite time series.

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ABSTRACT

In this paper we prove the existence of extreme value laws for dynamical systems perturbed by the instrument-like-error, also called observational noise. An orbit perturbed with observational noise mimics the behavior of an instrumentally recorded time series. Instrument characteristics – defined as precision and accuracy – act both by truncating and randomly displacing the real value of a measured observable. Here we analyze both these effects from a theoretical and a numerical point of view. First we show that classical extreme value laws can be found for orbits of dynamical systems perturbed with observational noise. Then we present numerical experiments to support the theoretical findings and give an indication of the order of magnitude of the instrumental perturbations which cause relevant deviations from the extreme value laws observed in deterministic dynamical systems. Finally, we show that the observational noise preserves the structure of the deterministic attractor. This goes against the common assumption that random transformations cause the orbits asymptotically fill the ambient space with a loss of information about the fractal structure of the attractor.

1. Introduction

In two previous works [1,2], we investigated the persistence of Extreme Value Laws (EVLs) whenever a dynamical system is perturbed throughout random transformations. We considered an i.i.d. stochastic process $(\omega_k)_{k\in N}$ with values in the measurable space Q_ε and with probability distribution θ_ε. After associating to each $\omega \in Q_\varepsilon$ a map T_ω acting on the measurable space Ω into itself, we considered the random orbit starting from the point x and generated by the realization $\underline{\omega}_n = (\omega_1, \omega_2, \ldots, \omega_n)$:

$$T_{\underline{\omega}_n} := T_{\omega_n} \circ \cdots \circ T_{\omega_1}(x).$$

In this setting the transformations T_ω are taken close to each other and the suitably rescaled scalar parameter ε is the strength of such a distance. We could therefore define a Markov process $\mathcal{X}_\varepsilon$ on Ω with transition function

$$P(x, A) = \int_{Q_\varepsilon} \mathbf{1}_A(T_\omega(x))d\theta_\varepsilon(\omega), \tag{1.1}$$

where $A \in \Omega$ is a measurable set, $x \in \Omega$ and $\mathbf{1}_A$ is the indicator function of a set A. We recall that a probability measure μ_ε is called a *stationary measure* if for any measurable A we have:

$$\mu_\varepsilon(A) = \int_\Omega P_\varepsilon(x, A)d\mu_\varepsilon(x).$$

Moreover, we call it an absolutely continuous stationary measure (*acsm*), if it has a density with respect to the Lebesgue measure whenever Ω is a metric space.

In this work we consider a different type of perturbation, the observational noise, which consists in replacing the orbit of the

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point $x \in \Omega$ at time i, namely $T^i x$, with $T^i x + \omega_i$. There are several physical motivations to investigate the behavior of this kind of perturbation. In fact, as Lalley and Noble wrote in [3]:

"*...In this model our observations take the form $y_i = T^i x + \omega_i$, where ω_i are independent, mean zero random vectors. In contrast with the dynamical noise model (e.g.; the random transformations), the noise does not interact with the dynamics: the deterministic character of the system, and its long range dependence, are preserved beneath the noise. Due in part to this dependence, estimation in the observational noise model has not been broadly addressed by statisticians, though the model captures important features of many experimental situations.*"

Judd [4], quoted in [5], also pointed out that:

"*...the reality is that many physical systems are indistinguishable from deterministic systems, there is no apparent small dynamic noise, and what is often attributed as such is in fact model error.*"

Moreover, a system contaminated by the observational noise raises the natural and practical question whether it would be possible to recover the original signal, in our case the deterministic orbit $\{T^i x\}_{i\geq 1}$. In the last years a few techniques have been proposed for such a noise reduction [6]: we remind here the remarkable Schreiber–Lalley method [7–10], which provides a very consistent algorithm to perform the noise reduction when the underlying deterministic dynamical system has strong hyperbolic properties. Another interesting work shows that in the computation of some statistical quantities, the dynamical noise corresponding to random transformations could be considered as an observational noise with the Cauchy distribution [11]. Finally, the paper [12] proves concentration inequalities for systems perturbed by observational noise.

The present work tries to re-frame the previous findings in terms of extreme value theory (EVT) by adding a further motivation driven by the applicability of the whole EVT for dynamical systems to experimental data. It should be a general concern to check the role of instrument-like-perturbations before applying dynamical systems techniques to experimental datasets. In this sense, the dynamical systems considered in this paper share several properties with observed time series, as the observational noise acts exactly as a physical instrument. The goal is to exploit the recent advancements of the EVT for dynamical systems to define in a more rigorous way the extremes of time series. A successful application of the theory presented in this paper to experimental datasets is given in [13], where temperature data are analyzed with the algorithmic procedure presented in Section 4.2. More specifically, our interest is to understand which way the results obtained on deterministic dynamical systems are altered by the addition of observational noise and in which cases one can recover classical EVLs. We start the discussion by summarizing the main findings of the EVT for dynamical systems.

The first rigorous mathematical approach to EVT in dynamical systems goes back to the pioneer paper by Collet [14]. Important contributions have successively been given in [15–17] and in [18]. Here we briefly recall the main findings deferring to the previous papers for the full demonstrations.

Let us consider a dynamical system $(\Omega, \mathcal{B}, \nu, T)$, where Ω is the invariant set in some manifold, usually $\mathbb{R}^d$, $\mathcal{B}$ is the Borel σ-algebra, $T : \Omega \to \Omega$ is a measurable map and ν a probability T-invariant Borel measure.

In order to adapt the EVT to dynamical systems, we follow [15]. We consider the stationary stochastic process $X_0, X_1, \ldots$ given by:

$$X_m(x) = w(\mathrm{dist}(T^m x, z)) \quad \forall m \in \mathbb{N}, \tag{1.2}$$

where 'dist' is a distance on the ambient space Ω, z is a given point and w is a suitable function which will be specified later. This particular functional form has been introduced first by Collet [14] and allows for a direct connection between recurrence properties around a point of the phase space z and the existence of EVLs. The object of interest is the distribution of $\mathbb{P}(M_m \leq u_m)$, where $M_m := \max\{X_0, \ldots, X_{m-1}\}$; we have an EVL for M_m if there is a non-degenerate distribution function $H : \mathbb{R} \to [0, 1]$ with $H(0) = 0$ and, for every $\tau > 0$, there exists a sequence of levels $u_m = u_m(\tau)$, $m = 1, 2, \ldots$, such that

$$m\,\mathbb{P}(X_0 > u_m) \to \tau, \quad \text{as } m \to \infty, \tag{1.3}$$

and for which the following limit holds:

$$\mathbb{P}(M_m \leq u_m) \to 1 - H(\tau), \quad \text{as } m \to \infty.$$

The motivation for using a normalizing sequence u_m satisfying (1.3) comes from the case when $X_0, X_1, \ldots$ are independent and identically distributed (i.i.d.). In this setting, it is clear that $\mathbb{P}(M_m \leq u) = (F(u))^m$, being $F(u)$ the cumulative distribution function for the variable u. Hence, condition (1.3) implies that

$$\mathbb{P}(M_m \leq u_m) = (1 - \mathbb{P}(X_0 > u_m))^m \sim \left(1 - \frac{\tau}{m}\right)^m \to \mathrm{e}^{-\tau},$$

as $m \to \infty$. Note that in this case $H(\tau) = 1 - \mathrm{e}^{-\tau}$ is the standard exponential distribution function. By choosing the sequence $u_m = u_m(y)$ as one parameter families like $u_m = y/a_m + b_m$, where $y \in \mathbb{R}$ and $a_m > 0$, for all $m \in \mathbb{N}$ and w as above, we have $\mathbb{P}(a_m(M_m - b_m) \leq y) \to G(y)$ whenever the variables X_i are i.i.d., if for some constants $a_m > 0$, b_m. When the convergence occurs at continuity points of G (G is non-degenerate) then G_m converges to one of the three EVLs rewritable in terms of the Generalized Extreme Value (GEV) distribution as:

$$G(y; \kappa) = \exp\left\{[1 + \kappa y]^{-1/\kappa}\right\}. \tag{1.4}$$

Here $\kappa \in \mathbb{R}$ is the shape parameter also called the tail index: when $\kappa \to 0$, the distribution corresponds to a Gumbel EVL; when the tail index is positive, it corresponds to a Fréchet EVL; when κ is negative, it corresponds to a Weibull EVL. The EVL obtained depends on the kind of observable chosen. In particular, in [14,15] the authors have shown that, once taken the observable:

$$w(y) = -\log(y), \tag{1.5}$$

one gets a Gumbel EVL, here $y = \mathrm{dist}(T^m x, z)$. In the next section we prove the existence of Gumbel law for the maps perturbed with observational noise. It is in fact possible to introduce other observables than the one specified above in order to get convergence towards Fréchet and Weibull EVLs. However, for any choice different from $w(y) = -\log(y)$, the tail index can be written in terms of the local dimension (see Eqs. (4.2)–(4.4) in [19]). For a sequence $(u_m)_{m\in\mathbb{N}}$ satisfying (1.3) we define:

$$U_m := \{X_0 > u_m\}. \tag{1.6}$$

When $X_0, X_1, X_2, \ldots$ are not independent, the standard exponential law still applies under some conditions on the dependence structure. These conditions are the following:

Condition ($D_2(u_m)$)**.** We say that $D_2(u_m)$ holds for the sequence $X_0, X_1, \ldots$ if for all ℓ, t and m,

$$\begin{aligned} &|\mathbb{P}(X_0 > u_m \cap \max\{X_t, \ldots, X_{t+\ell-1} \leq u_m\}) \\ &\quad - \mathbb{P}(X_0 > u_m)\mathbb{P}(M_\ell \leq u_m)| \leq \gamma(m, t), \end{aligned} \tag{1.7}$$

where $\gamma(m, t)$ is decreasing in t for each m and $m\gamma(m, t_m) \to 0$ when $m \to \infty$ for some sequence $t_m = o(m)$.

Now, let $(k_m)_{m\in\mathbb{N}}$ be a sequence of integers such that

$$k_m \to \infty \quad \text{and} \quad k_m t_m = o(m). \tag{1.8}$$