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Spatio-temporal oscillations in the Keller–Segel system with logistic growth

Shin-Ichiro Ei [a], Hirofumi Izuhara [b,*], Masayasu Mimura [b]

[a] Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

[b] Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, 4-21-1, Nakano, Nakano-ku, Tokyo 164-8525, Japan

HIGHLIGHTS

- The Keller–Segel system with logistic growth exhibits two types of oscillations.
- Different transitions of patterns depending on a certain parameter are observed.
- The origin of one of the oscillations is a relaxation oscillation.

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ABSTRACT

The Keller–Segel system with the logistic growth term is discussed from the spatio-temporal-oscillation point of view. This system exhibits two different types of spatio-temporal oscillations in certain distinct parameter regimes. In this paper, we study the difference between the two types of spatio-temporal oscillations. In particular, the characteristic properties of the behaviors become clear in a limiting system when a certain parameter value tends to zero. Moreover, we demonstrate that the onset of one of the spatio-temporal oscillatory patterns is an infinite-dimensional relaxation oscillation that consists of slow and fast dynamics.

1. Introduction

Several types of pattern phenomena can be observed in nature. For the theoretical understanding of such pattern phenomena, PDE models have been proposed and analyzed ([1] and the references therein). Among these models, Keller–Segel-type systems with growth have been proposed in order to describe the pigmentation pattern formation on snakes, pattern formation in bacterial colonies of *Salmonella typhimurium*, tumor cell invasion of tissue, and so on (for instance, [2–9]). A general form of the systems is as follows:

$$\begin{cases} u_t = D_u \Delta u - \operatorname{div}(u \nabla \chi(v)) + f(u), \\ v_t = D_v \Delta v + g(u, v), \end{cases}$$

where u and v indicate the cell density with diffusion rate D_u and the concentration of a chemical substance with diffusion rate D_v, called chemoattractant, respectively. The second term of the right hand side in the equation for u implies that the cells directly migrate to the higher concentration region of the chemoattractant in the macroscopic scale, which process is called chemotaxis. The function $\chi(v)$ included in the chemotaxis term denotes a sensitivity function for the chemoattractant. For the explanation of $\chi(v)$, we refer to the papers [10,11] and the references therein. Typical examples of the sensitivity function are $\chi(v) = \frac{\alpha v}{1+\beta v}$ (Michaelis–Menten receptor kinetics), $\chi(v) = \alpha v$ (direct measurement), and so on. The third term $f(u)$ in the equation for u represents the proliferation of cells. A well known example of the growth term is the logistic growth $f(u) = \delta u(1 - \frac{u}{K})$, where δ and K denote the growth rate and the carrying capacity of u, respectively. Further, $g(u, v)$ indicates the production and degradation of the chemoattractant. The simplest form is $g(u, v) = au - bv$, where both the production and the degradation of the chemoattractant occur at constant rates.

* Correspondence to: Faculty of Engineering, University of Miyazaki, 1-1, Gakuen Kibanadai Nishi, Miyazaki 889-2192, Japan. Tel.: +81 985 58 7384.
E-mail address: izuhara@cc.miyazaki-u.ac.jp (H. Izuhara).

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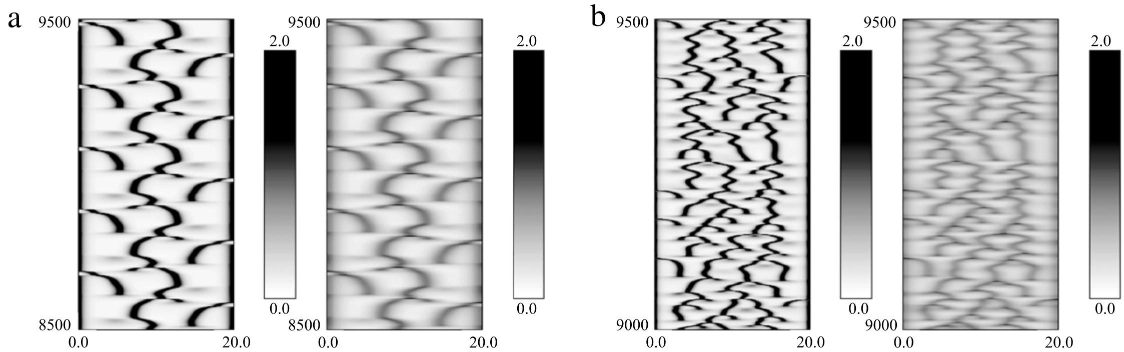

Fig. 1. One-dimensional spatio-temporal oscillatory patterns arising in the Keller–Segel-growth system (1.3), (1.4), and (1.5). The left and right figures correspond respectively to u and v, where the color scale indicates that white color corresponds to zero and black color corresponds to values greater than two. The horizontal and vertical axes in the figure indicate space x and time t, respectively. (a) A regular spatio-temporal oscillation when the parameter values are $D = 1.2$, $\alpha = 7$, $\delta = 0.4$, and $L = 20$. (b) An irregular spatio-temporal oscillation when the parameter values are $D = 0.5$, $\alpha = 6$, $\delta = 1$, and $L = 20$.

In this paper, we deal with the following simplest Keller–Segel system with growth:

$$\begin{cases} u_t = D_u \Delta u - \alpha \mathrm{div}(u \nabla v) + \delta u \left(1 - \dfrac{u}{K}\right), \\ v_t = D_v \Delta v + au - bv, \end{cases} \tag{1.1}$$

where all parameters are positive constants. By using the following rescaling for nondimensionalization:

$$x^* = \sqrt{\frac{b}{D_v}}x, \qquad t^* = bt, \qquad u^* = \frac{u}{K}, \qquad v^* = \frac{bv}{aK},$$

$$D^* = \frac{D_u}{D_v}, \qquad \alpha^* = \frac{\alpha a K}{D_v b}, \qquad \delta^* = \frac{\delta}{b},$$

(1.1) is written as

$$\begin{cases} u_t = D \Delta u - \alpha \mathrm{div}(u \nabla v) + \delta u(1-u), \\ v_t = \Delta v + u - v, \end{cases} \tag{1.2}$$

where asterisks are dropped. We call (1.2) the Keller–Segel-growth system. Here, we consider (1.2) in one space dimension

$$\begin{cases} u_t = Du_{xx} - \alpha(uv_x)_x \\ \quad + \delta u(1-u), \qquad t > 0, \ x \in I := (0, L) \\ v_t = v_{xx} + u - v, \end{cases} \tag{1.3}$$

with the Neumann boundary conditions

$$\begin{cases} u_x(t, 0) = u_x(t, L) = 0, \\ v_x(t, 0) = v_x(t, L) = 0, \end{cases} \quad t > 0 \tag{1.4}$$

and the initial conditions

$$\begin{cases} u(0, x) = u_0(x), \\ v(0, x) = v_0(x), \end{cases} \quad x \in I. \tag{1.5}$$

It is obviously known that $(\overline{u}, \overline{v}) = (1, 1)$ is a spatially homogeneous stationary solution of (1.3) and (1.4). It is a natural first step to study how the solution's stability changes depending on the parameters in (1.3). For instance, we assume α as a free parameter and fix δ suitably. When $\alpha = 0$, the equation for u is simply reduced to the Fisher–KPP equation [12]. Consequently, it is well known that $u = 1$ is the stable homogeneous stationary solution. Therefore, when α is small, we note that $(1, 1)$ is stable. On the other hand, when α is suitably large, we can expect that $(1, 1)$ is destabilized due to the strength of the chemotaxis effect. This is called the chemotaxis-induced instability. Conversely, when we assume δ as a free parameter and fix α appropriately, the stationary solution $(1, 1)$ should be stable for large δ values because the growth effect becomes dominant, that is, u approaches 1 everywhere. However, when δ decreases, the homogeneous stationary solution $(1, 1)$ is destabilized by virtue of the aggregation effect because (1.3) in the extreme case of $\delta = 0$ is reduced to the classical Keller–Segel system [13,1]. Therefore, we note that α and δ are important parameters contributing to the occurrence of instability of the spatially homogeneous stationary solution $(1, 1)$.

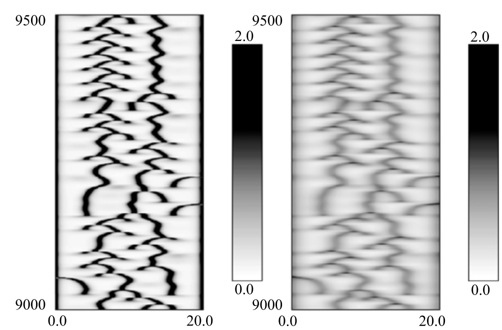

Fig. 2. One-dimensional irregular spatio-temporal oscillation arising in the Murray–Myerscough system (1.6), (1.4), and (1.5). The parameter values are $D = 1$, $\alpha = 10$, $\beta = 0.1$, $\delta = 1$, and $L = 20$.

Recently, it has been reported by Painter and Hillen [14] that spatio-temporal oscillatory patterns are observed in (1.3), (1.4), and (1.5) in a certain parameter regime. They also observed irregular spatio-temporal oscillations, as shown in Fig. 1(b), and they conclude that such irregularity should be chaos by computing the Lyapunov exponent. Further, they conclude that a series of periodic solutions reaches chaos through the period-doubling cascade when a certain parameter is varied (see Figure 9 in [14]). This type of spatio-temporal oscillation is observed for a wide range of the parameter values and a wide class of chemotaxis systems (see for instance, [3,4,15,16]). In fact, one example is the following Murray–Myerscough system introduced in [8]

$$\begin{cases} u_t = Du_{xx} - \alpha\,(uv_x)_x + \delta u(1-u), \\ v_t = v_{xx} + \dfrac{u}{1+\beta u} - v, \end{cases} \qquad t > 0, \ x \in I \tag{1.6}$$

with zero-flux boundary conditions, which also generates an irregular spatio-temporal oscillation qualitatively similar to the one in Fig. 1(b) for appropriate parameter values, as shown in Fig. 2.

In this paper, we emphasize that there occurs another type of spatio-temporal oscillation in the Keller–Segel-growth system (1.3), (1.4), and (1.5), as shown in Fig. 3(a). It appears that the behavior of the spatio-temporal oscillatory pattern in Fig. 3(a) is drastically different from the one in Fig. 1. Such a pattern is also observed in the Murray–Myerscough system (1.6), (1.4), and (1.5), as shown in Fig. 3(b) and a related system [17]. Here, we note on the numerical scheme to solve (1.3) or (1.6) with (1.4) and (1.5) that all figures in the present paper are computed by the finite volume