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# Exponential synchronization of Kuramoto oscillators using spatially local coupling

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## HIGHLIGHTS

- The exponential synchronization of the Kuramoto model with spatially local coupling.
- A new novel approach without the linearization and perturbation method.
- Sufficient conditions for initial configurations leading to the exponential decay toward the completely synchronized states.
- Relations between the decay rates and eigenvalues of the graph Laplacian.

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## ABSTRACT

We study the generalized Kuramoto model of coupled phase oscillators with a finite size, and discuss the asymptotic complete phase–frequency synchronization. The generalized Kuramoto model has inherent difficulties in mathematical approaches that this model is governed by nonlinear equations and the Kuramoto oscillator is arbitrarily connected with the others. To overcome these mathematical barriers, many researchers have adopted a linearization of homogeneous solutions, and applied a perturbation method. However, we introduce a new method which just requires some conditions on the smallest and largest positive eigenvalues of the graph Laplacian, and directly compute the bounds of homogeneous solutions. Using this method, we present analytic results for the generalized Kuramoto model. More specifically, we give a few sufficient conditions for initial configurations leading to the exponential decay toward the completely synchronized states. Our sufficient conditions and decay rate depend on the coupling strength, the initial phase and natural frequency configurations, and the graph Laplacian, but the conditions are independent of the system size. Moreover, we estimate the time evolution of deviations for the phase and frequency, and show that the smallest and largest positive eigenvalues for the graph Laplacian affect the stability region and convergence rate for the synchronized states. Finally, we compare our analytic results with numerical simulations using a few examples.



## 1. Introduction

Synchronization is a key concept to the understanding of self-organization phenomena occurring in coupled oscillators of the dissipative type. This subject is observed in many natural, social, physical and biological systems, and has been found to have applications to a variety of fields. Especially, the collective behavior of limit-cycle oscillators appears in various biological phenomena such as flash of fireflies, chorus of crickets, synchronous firing of a cardiac pacemaker, and unification of brain signals (see [1–4]). As a result, the subject of synchronization is continuously calling for serious and systematic investigation, and has evolved as an independent field of scientific research.

In a mathematical point of view, the collective behavior of oscillators was pioneered by Winfree and Kuramoto [5–8] who took advantage of a simple kinetic theory on phase evolution. Since then, the dramatically increasing interests in synchronization have been pervading the study of nonlinear dynamical systems [9–12]. Among various models for synchronization phenomena, our main concern is the Kuramoto model which is a simple mean-field model of coupled oscillators with spatially local and sinusoidal coupling [1,13].

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The Kuramoto model consists of a population of $N$ oscillators for which dynamics is governed by the following equations:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N,$$

where $\theta_i$ is the phase of $i$th oscillator, $\omega_i$ is its natural frequency, $K > 0$ is the coupling strength between pairs of connected oscillators and $a_{ij}$ is the element of the coupling matrix. In particular, the traditional Kuramoto model is recovered by letting $a_{ij} = 1/N$ for all $i \neq j$ (all-to-all coupling).

There are two main perspectives to research the Kuramoto model. One is the synchronization phenomena of phases or frequencies with respect to time. The other is stability which is a local property of the vicinity of the synchronized solution. For the identical or non-identical Kuramoto oscillators with all-to-all coupling, synchronization estimates have been extensively studied analytically and numerically. Especially, Ermentrout [14] found a critical ratio between coupling strength and variance of natural frequencies for which oscillators become phase-locked regardless of the number of oscillators. The stability of phase-locking state was established by van Hemmen and Wreszinski [15] for the strong coupling using the Lyapunov functional approach, and Jadbabaie et al. [16] also considered the stability analysis for coupled nonlinear oscillators with arbitrary connectivity and derived a computable bound for the critical coupling constant using the spectral graph theory. Recently, Ha et al. [17] found a necessary condition for initial phase configurations that the complete frequency synchronization occurs exponentially fast.

In this paper, we consider the generalized Kuramoto model as follows:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} \frac{L_{ij}}{d_i} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N,$$

where $L_{ij}$ are the elements of the connectivity matrix and $d_i$ are the degree of $i$th oscillator. We say that $L_{ij}/d_i$ is a weighted coupling. This form has been used to solve the paradox of heterogeneity that the heterogeneity in the degree distribution may suppress synchronization in networks of oscillators coupled symmetrically with uniform coupling strength (for details, see [18] and the references therein). Especially, Motter et al. [18] referred to the stability of the completely (or fully) synchronized state of this model.

The traditional Kuramoto system is governed by nonlinear equations. However it is based on a complete graph and used to linearize around synchronized states. These are advantages to analytical studies for the complete synchronization. On the other hand, a generalized Kuramoto model is given by arbitrarily connected networks (graph Laplacian) such as our case. Kalloniatis [19] studies the synchronization properties of the Kuramoto model of coupled phase oscillators on a general network. To show these, he used perturbation methods and the properties of eigenvalues for the graph Laplacian. In [20], to study synchronization of large interacting systems, Jost and Joy used the spectrum of the graph Laplacian and its relation to the stability properties of the spatially homogeneous solutions using linear stability analysis. Atay and Karabacak [21] presented a role of the largest eigenvalue to study the stability analysis of networks, both with and without time delays. In [22], Hagberg and Schult introduced that rewiring based on spectral properties of the graph Laplacian is effective at enabling synchronization.

In the above results, the main idea is a linearization of homogeneous solutions by using the spectral properties of the graph Laplacian, and then applying perturbation methods. One of the novelties of this paper is to perform a theoretic analysis for the asymptotic complete synchronization of the finite oscillators with arbitrary connectivity without any linearization procedure and perturbation method. Specifically, we provide sufficient conditions concerned with initial configurations and conductivity of oscillators, and show that the asymptotic complete synchronization is strongly affected by the graph structure and initial configurations. To prove these, our main strategy is to employ graph theory and derive differential inequalities for the weighted standard deviations of the phase and frequency configurations. Finally, we compute explicitly that the general Kuramoto oscillators exponentially decay to the completely synchronized estimates and the decay rate is bounded by the smallest and largest positive eigenvalues for the graph Laplacian, and we confirm these phenomena using a few examples.

The rest of this paper is divided into four sections after this introduction. In Section 2, we describe the generalized Kuramoto model, and present prior results on connectivity matrix. In Section 3, we prove that a complete phase and frequency synchronization for Kuramoto's oscillators occur exponentially fast. In Section 4, we present several numerical simulation results, and compare them with the analytical results in Section 3. Finally Section 5 is devoted to the summary of main results.

## 2. Preliminaries

In this section, we briefly introduce the general Kuramoto model, and obtain its equivalence equation and a few properties of eigenvalues for the graph Laplacian.

### 2.1. Model description

In the most popular version of the Kuramoto model, each of the oscillators $\theta_i$ is considered to have its own intrinsic natural frequency $\omega_i$, and each is coupled equally to the other oscillators. On the other hand, our concern is to investigate the dynamics of oscillators in the case that the Kuramoto model has a weighted interaction factor. The general Kuramoto model has the following governing equations:

$$\frac{d\theta_i}{dt} = \omega_i + K \sum_{j=1}^{N} \frac{L_{ij}}{d_i} \sin(\theta_j - \theta_i), \tag{1}$$

where $K$ is the coupling strength, $L_{ij}$ are the elements of the connectivity matrix and $d_i$ is the degree of the oscillator $\theta_i$, namely,

$$L_{ij} = L_{ji} = \begin{cases} 1, & \text{if } i \sim j, \\ 0, & \text{if not} \end{cases}$$

where the symbol "$\sim$" means that $\theta_i$ and $\theta_j$ are adjacent and $d_i = \sum_j L_{ij}$. Note that the general Kuramoto model has real analytic solution (see Hale's book [23] for details).

We finally show the complete phase and frequency synchronization of the general Kuramoto model as follows:

**Definition 2.1.** Let $\{\theta_i\}_{i=1}^N$ be the solutions to the system (1).

(i) The system $\{\theta_i\}_{i=1}^N$ has asymptotic complete phase synchronization if and only if the following condition holds.

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0, \quad i, j = 1, \ldots, N.$$

(ii) The system $\{\theta_i\}_{i=1}^N$ has asymptotic complete frequency synchronization if and only if the following condition holds.

$$\lim_{t\to\infty} |\theta_i'(t) - \theta_j'(t)| = 0, \quad i, j = 1, \ldots, N.$$