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The vanishing twist in the restricted three-body problem

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HIGHLIGHTS

- We demonstrate the existence of twistless tori in the planar circular restricted three-body problem.
- The associated reconnection bifurcations and meandering curves are found.
- The Birkhoff normal form at the Lagrangian triangular equilibrium is calculated to eighth order.
- Numerically integrated Poincare sections agree well with predictions made from the truncated integrable normal form.

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ABSTRACT

This paper demonstrates the existence of twistless tori and the associated reconnection bifurcations and meandering curves in the planar circular restricted three-body problem. Near the Lagrangian equilibrium $\mathcal{L}_4$ a twistless torus is created near the tripling bifurcation of the short period family. Decreasing the mass ratio leads to twistless bifurcations which are particularly prominent for rotation numbers 3/10 and 2/7. This scenario is studied by numerically integrating the regularised Hamiltonian flow, and finding rotation numbers of invariant curves in a two-dimensional Poincaré map.

To corroborate the numerical results the Birkhoff normal form at $\mathcal{L}_4$ is calculated to eighth order. Truncating at this order gives an integrable system, and the rotation numbers obtained from the Birkhoff normal form agree well with the numerical results. A global overview for the mass ratio $\mu \in (\mu_4, \mu_3)$ is presented by showing lines of constant energy and constant rotation number in action space.

1. Introduction

The planar circular restricted three-body problem (PCR3BP) describes the dynamics of a body of negligible mass (the *test particle*) travelling in the gravitational field of two bodies with masses m_1 and m_2 (the *primaries*). The primaries are assumed to have circular orbits, and all three bodies are restricted to a single plane. In the rotating frame of reference with the primaries fixed, there are five equilibria for the test particle, known as the *Lagrange points*. In particular we are concerned with the triangular Lagrange equilibria $\mathcal{L}_4$ and $\mathcal{L}_5$ (which occur at the third point of equilateral triangles with the primaries). In celestial mechanics bodies whose orbit remains close to these Lagrange points are called Trojans, in particular in the Sun–Jupiter system. It is well-known (see for example [1]) that if $\mu = \frac{m_2}{m_1+m_2} \le \mu_1 = 0.5(1-\sqrt{69}/9) \approx 0.0385$, then $\mathcal{L}_4$ and $\mathcal{L}_5$ are elliptic; for all parameter values considered in this paper, this condition is met. The monographs [2,3] contain a wealth of information on the problem. Here we are interested in invariant tori and their bifurcations near $\mathcal{L}_4$. The dynamics near $\mathcal{L}_4$ has also been studied extensively; see, e.g., [4–7].

At the triangular Lagrange equilibria for $\mu < \mu_1$ the linearised Hamiltonian has two pairs of pure imaginary eigenvalues $\pm i\omega_s$ and $\pm i\omega_l$ (corresponding to the short and long period families respectively). The mass ratio μ for which $\omega_s/\omega_l = r > 1$ is denoted by μ_r; see, e.g., [1]. According to the Lyapunov centre theorem, see, e.g., [1], the short period family always exists for $\mu < \mu_1$ and has period $2\pi/\omega_s$ when approaching the origin. When r is not an integer then for $\mu = \mu_r$ the long period family exists and has period $2\pi/\omega_l$ when approaching the origin. For the range we are considering here $\mu \in (\mu_3, \mu_4)$ both the long and the short family exist. This range includes $\mu = 0.01215$ for the Earth–Moon system. See Table 1 for some relevant values of μ_r.

The computation of the Birkhoff normal form at $\mathcal{L}_4$ was first performed by Deprit and Deprit-Bartholomé in 1967 [8] up to order 4 (degree 2 in actions). They found the special value $\mu_c \approx 0.0109$ for which the iso-energetic non-degeneracy condition of the KAM theorem does not hold at $\mathcal{L}_4$, and hence potentially stability could have been lost since the Hamiltonian is not definite at $\mathcal{L}_4$. Meyer

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Table 1
μ_r for rational values of r between the 1/4 and 1/3 resonances, to five decimal places. μ_c is the value where the twist vanishes at $\mathcal{L}_4$. Rotation number 5/16 approximately occurs for the Earth–Moon system.

r	4	$\frac{11}{3}$	$\frac{7}{2}$	c	$\frac{10}{3}$	$\approx\frac{16}{5}$	3
μ_r	0.00827	0.00964	0.01045	0.01091	0.01135	0.01215	0.01351

and Schmidt in 1986 [9] pushed the computation to order 6 using computer algebra. With these results they established the stability of $\mathcal{L}_4$ by KAM theory for all parameters in the elliptic range $\mu < \mu_1$, except at lowest order resonances μ_2 and μ_3, but including μ_c.

The vanishing of twist at the equilibrium point is not only a threat to KAM stability, but also signals the creation of a twistless torus nearby. Standard iso-energetic KAM theory fails near a twistless torus. There are KAM theorems with weaker non-degeneracy assumptions that fix this problem [10]. But in any case, the passage of the twistless torus through a rational rotation number under variation of a parameter (e.g. the energy) gives rise to twistless bifurcations. Such bifurcations have first been studied in area preserving maps by Howard [11,12], and a normal form was derived in [13]. Results on break-up of twistless curve and a review of the literature can be found in [14].

To analyse twistless bifurcations in an integrable or near-integrable map the most useful presentation of the dynamics of the map is in the form of (approximate) action-angle variables (I, θ), so that the unperturbed dynamics is $I' = I$ and $\theta' = \theta + W(I)$. At a twistless torus $I = I^*$ by definition the rotation number W has a critical point, $W'(I^*) = 0$. When the critical value $W(I^*)$ passes through a rational value for a typically perturbed map a twistless bifurcation occurs. The paper [15] on generic twistless bifurcations demonstrates that for a family of maps with a fixed point with rotation numbers in [1/6, 1/2] the twist of the fixed point must vanish for some $W \neq 1/3$. The vanishing twist at the origin corresponds to a critical point of W at the origin, i.e. $W'(0) = 0$. When the critical point of W moves to positive I under variation of a parameter a twistless curve is created. Another source of a twistless curve may be a saddle-centre bifurcation, as shown in [16], where the universal features of this bifurcation were also studied. For example, in the area preserving Hénon map a twistless curve is created in a saddle-centre bifurcation of a period three orbit near a fixed point with rotation number 1/3 [16].

Twistless bifurcations shown to be generic in 1-parameter families of area preserving maps [15] naturally also appear in Hamiltonian flows, simply because a Poincaré section of the flow at fixed energy gives a 2-dimensional area preserving map. Thus we should expect to see twistless bifurcations in a Hamiltonian flow, since the energy E may serve as the family parameter. So it is not a surprise to find twistless bifurcations in the restricted problem along 1-dimensional lines in parameter space of E and μ. Looking for twistless bifurcations, however, is a bit like looking for a needle in a haystack, since they often occur in very small regions in parameter space. Simó and Stuchi [17] were the first to demonstrate the existence of meandering twistless curves in Hill's problem, a variant of the PCR3BP where the distance between the two primaries is made infinite. The existence of twistless bifurcations in Hill's problem certainly implies the existence of such bifurcations in the PCR3BP. In the present work we show that twistless bifurcations occur not only near Hill's orbits, but also near the Lagrangian equilateral equilibrium point $\mathcal{L}_4$, where the approximation for Hill's problem is not valid.

The structure of the paper is as follows. The Hamiltonian and its regularisation is reviewed in the next section. In Section 3 we present numerically computed Poincaré sections and the corresponding numerically computed W curves. Our numerically computed results show that there are prominent twistless bifurcations for $W = 3/10$ and $W = 2/7$ along one-parameter families in (E, μ) space. These occur in the island of stability in the Poincaré section that has the short period orbit at its centre. The line of twistless bifurcations in parameter space traces closely the lines where the short period orbit has rotation number 3/10 and 2/7, respectively. Then we show that the numerical results can be analytically explained using a high order non-resonant Birkhoff normal form at the equilibrium $\mathcal{L}_4$. The presence of an equilibrium point is a feature of the flow that cannot be present in a family of maps. It serves as an anchor for the computation of an analytic approximation from which an approximation to $W(I)$ can be found. The normal form reveals the full complexity of the bifurcations of invariant tori around the short period family for $\mu \in (\mu_4, \mu_3)$ in the PCR3BP.

2. The planar circular restricted three-body problem

We construct the Hamiltonian describing the dynamics of the test particle as in [18]:

$$\mathscr{H}(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + p_x y - p_y\left(x + \mu - \frac{1}{2}\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} + s, \tag{1}$$

$$r_1 = \sqrt{\left(x + \frac{1}{2}\right)^2 + y^2}, \qquad r_2 = \sqrt{\left(x - \frac{1}{2}\right)^2 + y^2}, \qquad s = \frac{3 + \mu(\mu - 1)}{2}. \tag{2}$$

The Hamiltonians are normalised by s so that the energy of a stationary test particle at $\mathcal{L}_4$ is $E = 0$. The Hamiltonian has been scaled so the total mass of the primaries is 1 unit, and the distance between the primaries is also 1 unit. We take a rotating frame of reference, with the primaries fixed; the primary of mass μ at $(-1/2, 0)$ and $1 - \mu$ at $(+1/2, 0)$ for $\mu \in (0, 1/2]$, the *mass ratio*. The other parameter of the system is $E = \mathscr{H}$, the constant value of the Hamiltonian. Our E is related to the usual *Jacobi integral* $C = -2(E - s)$.

We then introduce complex coordinates $z = x + iy \in \mathbb{C}$, $p_z = p_x - ip_y \in \mathbb{C}$. Now the transformed Hamiltonian is

$$\mathscr{H}(z, p_z) = \frac{1}{2}|p_z|^2 + \operatorname{Im}\left((z + \frac{1}{2} - \mu)p_z\right) - \frac{1-\mu}{\left|z + \frac{1}{2}\right|} - \frac{\mu}{\left|z - \frac{1}{2}\right|} + s. \tag{3}$$

To regularise and remove the singularities at $z = \pm 1/2$, we take the Thiele transformation as described in [19–21]:

$$z = \frac{1}{2}\cos w, \qquad p_z = \frac{-2p_w}{\sin w}. \tag{4}$$

The new Hamiltonian is

$$\mathscr{H}(w, p_w) = 2\left|\frac{p_w}{\sin w}\right|^2 - \operatorname{Im}\left((\cos w + 1 - 2\mu)\frac{p_w}{\sin w}\right) - \frac{2-2\mu}{|\cos w + 1|} - \frac{2\mu}{|\cos w - 1|} + s. \tag{5}$$

Our final step is to take the Hamiltonian into extended phase space with time t and the fixed energy $E = \mathscr{H}(w, p_w)$ now