Journal of Geometry and Physics 62 (2012) 976–980

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Homogeneous Einstein metrics on $SU(n)$

A.H. Mujtaba

George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, TX 77843, USA

A R T I C L E   I N F O

Article history:
Received 17 October 2011
Accepted 26 January 2012
Available online 2 February 2012

Keywords:
Einstein metrics
SU(n)
Lie algebras
Manifolds

A B S T R A C T

It is known that every compact simple Lie group admits a bi-invariant homogeneous Einstein metric. In this paper we use two ansatz to probe the existence of additional inequivalent Einstein metrics on the Lie group $SU(n)$ for arbitrary n. We provide an explicit construction of $(2k + 1)$ inequivalent Einstein metrics on $SU(2k)$ and $2k$ inequivalent Einstein metrics on $SU(2k + 1)$.

1. Introduction

The Einstein equation $R_{\mu\nu} = \lambda g_{\mu\nu}$, which defines an Einstein metric, constrains the Ricci and through it the Riemann curvature tensor. In general in d dimensions the Ricci tensor has $\frac{1}{2}d(d+1)$ algebraically independent components while the Riemann Curvature Tensor has $\frac{1}{12}d^2(d^2 - 1)$ algebraically independent components [1]. So as d increases fewer constraints are placed on the curvature of the metric. In fact for $d \geq 4$ the number of independent components of the Ricci tensor is less than that of the Riemann tensor and the gap widens as d increases.

These considerations lead one to expect that as the number of dimensions d increases the number of (inequivalent) Einstein metrics should also increase [2]. In this paper we search for some of these increasing number of Einstein metrics on $SU(n)$ group manifolds.

The number of independent components of an $n \times n$ unitary matrix with unit-determinant are $n^2 - 1$ which means that the Lie group $SU(n)$ and its associated manifold has dimension $d = n^2 - 1$. Consequently the number of possible Einstein metrics on the $SU(n)$ group manifold increases rapidly with increasing n.

1.1. Construction of metrics

We choose to work with the vielbiens σ_a (1-forms) coupled with a metric g_{ab} (with constant components that do not vary with the parameters of the Lie-Algebra) [3]. Given a Lie group G with generators T_a, if $g \in G$ then the left-invariant 1-forms σ^a are given by

$$g^{-1}dg = \sigma^a T_a. \tag{1.1}$$

The general metric on the group manifold written in terms of the 1-forms is [3]

$$ds^2 = g_{ab}\,\sigma^a \sigma^b. \tag{1.2}$$

With the 1-forms σ^a defined, our task is to find metrics g_{ab} such that the metric ds^2 (as defined in (1.2)) is Einstein.

E-mail address: abid.naqvi83@gmail.com.

0393-0440/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2012.01.011

1.2. Additional Einstein metrics

Every simply compact Lie group admits a bi-invariant metric of the form $\mathrm{tr}(g^{-1}dg)^2$ which in a suitable choice of basis for the generators T_a can be expressed as

$$ds^2 = c\,\sigma^a\sigma_a, \tag{1.3}$$

where c is a constant [2]. This corresponds to $g_{ab} = c\,\delta_{ab}$.

D'Atri and Ziller [4] have shown that every simple compact Lie group, with the exception of $SU(2)$ and $SO(3)$, admits at least one additional homogeneous Einstein metric. These additional Einstein metrics, though not bi-invariant, are still invariant under the transitive G action (we have chosen to preserve the full G_L) [2].

In particular cases, homogeneous Einstein metrics have been shown to exist in addition to the bi-invariant and D'Atri and Ziller cases. Six inequivalent homogeneous Einstein metrics have been found explicitly for the exceptional group G_2 [5]; $(3k-4)$ and $(3k-3)$ inequivalent Einstein metrics on $SO(2k)$ and $SO(2k+1)$ respectively [2], and a considerable body of work in part concerning Einstein metrics on $SO(n)$, $Sp(n)$ [6–10]. These successes motivate the search for additional inequivalent Einstein metrics on $SU(n)$.

1.3. Inequivalence of Einstein metrics

When searching for new Einstein metrics one must determine whether a newly found candidate is truly "new" or if it is equivalent to an already known metric, possibly by a change of basis. A standard technique for evaluating this possibility is to calculate some dimensionless invariant quantity which is constructed from the metric and its curvature. We choose to use [2,5]:

$$I_1 = R_{abcd}\,R^{abcd}\,\lambda^{-2} = |\mathrm{Riem}|^2\,\lambda^{-2} \tag{1.4}$$

where $R_{ab} = \lambda g_{ab}$.

For any two Einstein metrics one calculates the value of the invariant. If the calculated values are unequal then the two Einstein metrics are clearly inequivalent. If the calculated values are equal the two metrics are likely equivalent but more investigation is required to prove it conclusively [5].

2. Metrics on $SU(n)$

We choose to construct and manipulate the metrics on $SU(n)$ using the left-invariant 1-forms ${L_A}^B$ where $1 \le A \le n$. These have the property ${L_A}^{B\dagger} = {L_B}^A$ and obey the algebra [5]

$$d{L_A}^B = i\,{L_A}^C \wedge {L_C}^B. \tag{2.1}$$

The total number of possible metrics is large given the freedom to construct the 1-forms σ^a (from the ${L_A}^B$) as well as the metric g_{ab}. In our search for Einstein metrics on $SU(n)$ we choose to study certain classes of ansatz. These take the form of schemes for the construction of Hermitian traceless 1-forms from the ${L_A}^B$ as well as particular choices for the metric g_{ab}.

2.1. Scheme 1

2.1.1. The generators

The scheme consists of the construction of n^2-1 traceless Hermitian 1-forms K_i from the ${L_A}^B$. Let $m \equiv n(n-1)/2$. We begin by constructing $n(n-1)/2$ "traceless Hermitian" 1-forms of the form

$$K_i = {L_A}^B + {L_B}^A \tag{2.2}$$

where $A \neq B$, for example $K_1 = {L_1}^2 + {L_2}^1$. The next $n(n-1)/2$ 1-forms will be taken to be

$$K_{m+i} = i\,({L_A}^B - {L_B}^A) \tag{2.3}$$

where $A \neq B$.

Since the 1-forms K_i are obtained by taking linear combinations of the ${L_A}^B$ one can describe the construction in the language of matrices. Let $\vec{l}$ be the vector with entries $l_i = {L_i}^i$ and $\vec{k}$ have entries $k_i = K_{2m+i}$ for $1 \le i \le n$ then[1]:

$$\vec{k} = \mathbf{P}\,\mathbf{Q}\,\vec{l} \tag{2.4}$$

[1] Note that the last K_i so defined is K_{n^2} which is not a 1-form of $SU(n)$ and has non-zero trace to boot (it corresponds to the unit matrix, that is the generator of $\mathfrak{u}(1)$).