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Algebraic integrability conditions for Killing tensors on constant sectional curvature manifolds

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ABSTRACT

We use an isomorphism between the space of valence two Killing tensors on an n-dimensional constant sectional curvature manifold and the irreducible $GL(n+1)$-representation space of algebraic curvature tensors in order to translate the Nijenhuis integrability conditions for a Killing tensor into purely algebraic integrability conditions for the corresponding algebraic curvature tensor, resulting in two simple algebraic equations of degree two and three. As a first application of this we construct a new family of integrable Killing tensors.

1. Introduction

1.1. Motivation

Besides the Euler–Lagrange formalism and the Hamilton formalism, the Hamilton–Jacobi equation is one of the three fundamental reformulations of classical Newtonian mechanics and has widespread applications in physics as well as mathematics, ranging from classical mechanics over optics and semi-classical quantum mechanics to Riemannian geometry. In many cases this first-order non-linear partial differential equation can be solved by a separation of variables after finding appropriate coordinates. The theory of separable coordinates and their classification has a long history, marked by the works of a number of prominent mathematicians: Bôcher [1], Stäckel [2], Levi-Civita [3], Darboux [4], Eisenhart [5], Kalnins and Miller [6]. For a review of recent developments, see [7] and the references therein.

The Hamilton–Jacobi equation separates in a given system of orthogonal coordinates if and only if there exists an integrable valence two Killing tensor field with simple eigenvalues whose eigenvectors are tangent to the coordinate lines and such that the potential satisfies a certain compatibility condition involving this Killing tensor [8]. Integrable Killing tensors are thus an important tool in the study of the separability of the Hamilton–Jacobi equation. But difficulties arise from the fact that the integrability condition is a complicated system of partial differential equations. In the present work we show how to cast these conditions into a simple algebraic form for constant sectional curvature manifolds.

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1.2. Method and result

Killing tensors form a linear space which is invariant under the pullback action of the manifold's isometry group. In other words, they constitute a representation space of the isometry group. McLenaghan et al. identified this representation in the case of constant sectional curvature manifolds as a certain irreducible representation of the general linear group [9]. More precisely, the isometric embeddings of the standard models of constant sectional curvature manifolds as hypersurfaces M in a Euclidean vector space (V, g) allow to write Killing tensors as restrictions of homogeneous polynomials on V, where the coefficients obey certain symmetry relations. This yields an explicit natural isomorphism between the space of valence two Killing tensors on M and the irreducible $\mathrm{GL}(V)$-representation space of algebraic curvature tensors on the ambient space V and is the starting point for the present work: If Killing tensors correspond naturally to algebraic curvature tensors, i.e. simple algebraic objects, then their integrability must be expressible as a purely algebraic condition on algebraic curvature tensors. To this aim we substitute the algebraic expression for a Killing tensor into the Nijenhuis integrability conditions and apply results from the representation theory for symmetric and general linear groups in order to derive the following simple algebraic integrability conditions, which describe the space of integrable Killing tensors as an algebraic variety:

Main Theorem 1. *A Killing tensor on a constant sectional curvature manifold M is integrable if and only if the associated algebraic curvature tensor R on V satisfies the following two conditions,*

$$\begin{array}{|c|}\hline a_2\\\hline b_2\\\hline c_2\\\hline d_2\\\hline\end{array}\;\bar g_{ij}R^i{}_{b_1\underline{a_2}\underline{b_2}}R^j{}_{d_1\underline{c_2}\underline{d_2}} = 0 \tag{1a}$$

$$\begin{array}{|c|}\hline a_2\\\hline b_2\\\hline c_2\\\hline d_2\\\hline\end{array}\;\begin{array}{|c|c|c|c|}\hline a_1 & b_1 & c_1 & d_1\\\hline\end{array}\;\bar g_{ij}\bar g_{kl}R^i{}_{b_1\underline{a_2}\underline{b_2}}R^j{}_{a_1}{}^k{}_{c_1}R^l{}_{d_1\underline{c_2}\underline{d_2}} = 0 \tag{1b}$$

where the operators on the left hand side are the Young symmetrisers for complete antisymmetrisation in the (underlined) indices a_2, b_2, c_2, d_2 respectively complete symmetrisation in the indices a_1, b_1, c_1, d_1. The tensor $\bar g$ denotes the inner product g on V in case M is not flat. Otherwise, i.e. if $M \subset V$ is a hyperplane, $\bar g$ is the (degenerated) pullback of g via the orthogonal projection $V \to M$.

Surprisingly, these conditions take a much simpler form than what would result from a straightforward substitution of the formula representing Killing tensors into the Nijenhuis integrability conditions. The latter would lead to projectors onto certain irreducible representations of the isometry group in place of the symmetrisation and antisymmetrisation operators above. Since the isometry group is a (pseudo-)orthogonal or Euclidean group, the explicit form of such projectors is quite complicated.

1.3. Advantages

This approach to integrability of Killing tensor fields on constant sectional curvature manifolds has a certain number of advantages.

The first and certainly the most important is, that we replace the Nijenhuis integrability conditions – a complicated non-linear system of partial differential equations for a tensor field on a manifold – by two simple algebraic equations for a tensor on a vector space. On the one hand this simplifies a numerical treatment considerably. Note that integrability can be checked by a simple evaluation of polynomials of degree two and three. On the other hand this opens the way for algebraic methods. The algebraic formulation for example allowed us to show that the third of the three Nijenhuis integrability conditions is redundant for Killing tensors on constant sectional curvature manifolds. In the special case of Euclidean 3-space, this result was already mentioned in a footnote of [10], stating that "Steve Czapor (private communication) has simplified the situation considerably. Using Gröbner basis theory, he has shown that (4.4a) and (4.4b) imply (4.4c), for any Killing tensor $\mathbf{K} \in \mathcal{K}^2(\mathbb{E}^3)$".[1]

For Euclidean 3-space a complete description of integrable Killing tensors has been obtained using computer algebra by Horwood et al. based on the prior knowledge of the separable coordinate webs, but a general solution of the integrability conditions has so far been considered intractable [10]. Our algebraic formulation has rendered this feasible at least in dimension three.[2] This goes beyond the scope of this article and will be the subject of a forthcoming paper [11].

In this context it is noteworthy that the first algebraic integrability condition can be recast into a variety of different forms. In terms of the curvature form $\Omega \in \mathrm{End}(V) \otimes \Lambda^2 V$ associated to the algebraic curvature tensor R, condition (1a)

1 (4.4) therein are the Nijenhuis integrability conditions, c.f. (7) here.

2 Note that dimension three of the manifold means dimension four of the ambient vector space.