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On the periodic orbits of perturbed Hooke Hamiltonian systems with three degrees of freedom

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Dedicated to a friend of both authors, Jorge Sotomayor, on the occasion of his 70th birthday

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ABSTRACT

We study periodic orbits of Hamiltonian differential systems with three degrees of freedom using the averaging theory. We have chosen the classical integrable Hamiltonian system with the Hooke potential and we study periodic orbits which bifurcate from the periodic orbits of the integrable system perturbed with a non-autonomous potential.

1. Introduction

In this paper we study the spatial motion of a particle of unitary mass under the action of a central force with Hamiltonian given by

$$H_0(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + V_0\left(\sqrt{x^2 + y^2 + z^2}\right),$$

perturbed by the Hamiltonian

$$H(x, y, z, p_x, p_y, p_z, t) = H_0(x, y, z, p_x, p_y, p_z) + \varepsilon V(t, x, y, z), \tag{1}$$

where ε is a small parameter and $V(t, x, y, z)$ is a perturbation of the potential eventually depending on the time t.

We consider a central force derived from a potential of the form

$$V_0\left(\sqrt{x^2 + y^2 + z^2}\right) = \pm(x^2 + y^2 + z^2)^{\alpha/2}, \tag{2}$$

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with α an integer. The Hamilton equations associated with Hamiltonian (1) are

$$\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{z} &= p_z, \\
\dot{p}_x &= -\partial \left(V_0 \left(\sqrt{x^2+y^2+z^2} \right) + \varepsilon V(t,x,y,z) \right) / \partial x, \\
\dot{p}_y &= -\partial \left(V_0 \left(\sqrt{x^2+y^2+z^2} \right) + \varepsilon V(t,x,y,z) \right) / \partial y, \\
\dot{p}_z &= -\partial \left(V_0 \left(\sqrt{x^2+y^2+z^2} \right) + \varepsilon V(t,x,y,z) \right) / \partial z,
\end{aligned} \tag{3}$$

where the dot denotes the derivative with respect to the time t. We shall apply the averaging theory for studying the periodic orbits of the Hamiltonian system (3).

The averaging method (see for instance [1]) gives a quantitative relation between the solutions of some non-autonomous periodic differential system and the solutions of its autonomous averaged differential system, and in particular allows us to study the periodic orbits of the non-autonomous periodic differential system as a function of the periodic orbits of the averaged one; see for more details [1–6] and, mainly, Section 2. Our aim is to apply the *averaging theory* to one class of Hamiltonian systems (3) for studying their periodic solutions. But the tools that we shall use are very general and can be applied to other classes of Hamiltonian or differential systems.

Of course there is a long tradition in studying the periodic orbits of the differential systems using the averaging method; see chapter 4 of [7], the book [1], chapter 11 of [6], the paper [8], and many others. In the paper [8] they use the averaging method of second order for studying the periodic orbits, but in that paper they studied a two-dimensional non-autonomous Hamiltonian system using the transformation to action–angle coordinates, and here we study six-dimensional non-autonomous Hamiltonian systems. On the other hand, this paper extends to Hamiltonian systems with three degrees of freedom some results of [9] where the authors studied periodic orbits of Hamiltonian systems with two degrees of freedom. We note that for applying the averaging method to Hamiltonian systems in general it is not necessary to write the unperturbed integrable Hamiltonian in action–angle coordinates.

The unique central forces coming from the central potentials of the form (2) for which all bounded orbits are periodic are the Hooke force and the Kepler force, which correspond to the potentials

$$k(x^2+y^2+z^2), \quad \text{and} \quad -\frac{k}{\sqrt{x^2+y^2+z^2}} \quad \text{with } k>0, \tag{4}$$

respectively. This result was proved by Bertrand in 1873; see [10].

We will apply the averaging theory to Hamiltonian systems (3) with the Hooke potential $V_0\left(\sqrt{x^2+y^2+z^2}\right) = k(x^2+y^2+z^2)$ for studying which periodic orbits of system (3) with $\varepsilon = 0$ can be continued to periodic orbits of the same system with $\varepsilon \neq 0$ sufficiently small. The periodic orbits for perturbed Kepler Hamiltonian systems with three degrees of freedom will be studied in a future article.

Since generically the periodic orbits of Hamiltonian systems live on cylinders full of periodic orbits, and every one of the periodic orbits of each of these cylinders belongs to a different level of the Hamiltonian (for more details see [11,12]), we shall apply the averaging theory described in Section 2 to Hamiltonian systems (3) with potential $V_0\left(\sqrt{x^2+y^2+z^2}\right) = k(x^2+y^2+z^2)$ restricted to a fixed level of the Hamiltonian. Working at a fixed level of the Hamiltonian is necessary for applying the averaging theory for studying periodic orbits (see Theorem 4), because the periodic orbits provided by the averaging must be isolated in the set of all periodic orbits.

Consider the spatial motion of a particle of unitary mass under the action of the Hooke potential $V_0\left(\sqrt{x^2+y^2+z^2}\right) = (x^2+y^2+z^2)/2$ perturbed by a non-autonomous potential $\varepsilon V(t,x,y,z)$ where ε is a small parameter and $V(t,x,y,z)$ is 2π-periodic in the variable t. In cartesian coordinates the Hamiltonian governing this motion is

$$H(x,y,z,p_x,p_y,p_z,t) = \frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right) + \frac{1}{2}\left(x^2+y^2+z^2\right) + \varepsilon V(t,x,y,z). \tag{5}$$

The corresponding Hamiltonian equations are

$$\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{z} &= p_z, \\
\dot{p}_x &= -x - \varepsilon \partial V(t,x,y,z)/\partial x, \\
\dot{p}_y &= -y - \varepsilon \partial V(t,x,y,z)/\partial y, \\
\dot{p}_z &= -z - \varepsilon \partial V(t,x,y,z)/\partial z.
\end{aligned} \tag{6}$$