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Characterizing killing vector fields of standard static space-times

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ABSTRACT

We provide a global characterization of the Killing vector fields of a standard static space-time by a system of partial differential equations. By studying this system, we determine all the Killing vector fields in the same framework when the Riemannian part is compact. Furthermore, we deal with the characterization of Killing vector fields with zero curl on a standard static space-time.

1. Introduction

The main concern of the current paper is to study the existence and characterization of Killing vector fields (KVF for short) of a standard static space-time (SSS-T for short).[1] Our approach partially follows that of Sánchez for Robertson–Walker space-times in [2], which is centrally supported by the structure of KVFs on warped products of pseudo-Riemannian manifolds, already obtained in the pioneering article of Bishop and O'Neill [3].

A *standard static space-time* (also called *globally static*, see [4]) is a Lorentzian warped product where the warping function is defined on a Riemannian manifold (called the *natural space* or *Riemannian part*) and acting on the negative definite metric on an open interval of real numbers (see Definition 3.2). This structure can be considered as a generalization of the Einstein static universe. In [5], it was shown that any static space-time[2] is *locally* isometric to a standard static one. There are many interesting and recent studies about several questions in SSS-Ts, see for instance [10–12,1,13–18] and references therein.

The existence of KVFs on pseudo-Riemannian manifolds was considered by many researchers (physicists [19] and mathematicians) from several points of view and by using different techniques. One of the first articles by Sánchez (i.e., [20]) is devoted to provide a review about these questions in the framework of Lorentzian geometry. In [2], Sánchez studied the structure of KVFs on a generalized Robertson–Walker space-time. He obtained necessary and sufficient conditions for a

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1 We would like to inform the reader that some of the results provided in this article were previously announced in the survey [1].

2 An n-dimensional space-time (M, g) is called *static* if there exists a nowhere vanishing time-like KVF X on M such that the distribution of $(n-1)$-plane orthogonal to X is integrable (see [6, Section 3.7] and also the general relativity texts [7–9]).

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vector field to be Killing on generalized Robertson–Walker space-times and gave a characterization of them as well as an explicit list for the globally hyperbolic case. In the recent survey [21] about general relativity, there appears a rich variety of questions where KVFs, stationary vector fields and black hole solutions play central roles.

Our first main result is about the characterization of KVFs on a SSS-T by a set of conditions similar to the conditions obtained by Carot and da Costa in [22] for the analogous *local* problem. Unfortunately, in their article (see [22, Section 4.2]) there are couple of computational mistakes that compromise the validity of their procedure but not their conclusions (see the Appendix). Here we apply an intrinsic notation (as in [2]) to obtain and provide global characterization conditions of KVFs on a SSS-T, obtaining as a side-product the correct relations corresponding to the procedure of Carot and da Costa.

In our second main result, we establish the central role of a particular over-determined system of partial differential equations involving the Hessian in the characterization of KVFs on SSS-Ts and studying these systems we completely characterize the KVFs of a SSS-T with compact Riemannian part. As an interesting application, we deal with the characterization of KVFs with *zero curl* (called here *non-rotating*) on a SSS-T.

The article is organized in the following way: in Section 2 we establish the main results. In Section 3 we give some useful preliminaries along the article. In Section 4 we prove the central results announced in Section 2 and other supplementary statements. In Section 5 we give some applications of the main results.

2. Description of main results

Throughout the article "*I will be an open real interval of the form* $I = (t_1, t_2)$ *where* $-\infty \leq t_1 < t_2 \leq \infty$". and "$(F, g_F)$ *will be a connected Riemannian manifold without boundary with* $\dim F = s$". We will denote the set of all strictly positive C^∞ functions defined on F by $C^\infty_{>0}(F)$.

Let $\mathbb{V}$ be an $\mathbb{R}$-vector space. For any subset S of $\mathbb{V}$, we use $\langle S\rangle$ to denote the $\mathbb{R}$-subspace of $\mathbb{V}$ generated by S. Briefly, if $\mathrm{x} \in \mathbb{V}$ we will write $\langle \mathrm{x}\rangle$ instead of $\langle\{\mathrm{x}\}\rangle$. Also, we will write $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$.

Suppose that $\mathscr{M}$ is a module over a ring $\mathbb{A}$ and $\mathscr{W} \subseteq \mathscr{M}$. If $\mathrm{v} \in \mathscr{M}$, then we will use the following notation $\mathrm{v} + \mathscr{W} = \{\mathrm{v} + W : W \in \mathscr{W}\}$.

Let $\mathscr{K}$ be the real Lie algebra of KVFs on (F, g_F). Given $\varphi, \psi \in C^\infty(F)$ we denote

$$\mathscr{K}_\varphi^\psi = \{K \in \mathscr{K} : K(\varphi) = \psi\}$$

and

$$\mathscr{K}_\varphi^{\langle\psi\rangle} = \{K \in \mathscr{K} : K(\varphi) \in \langle\psi\rangle\},$$

where the $\langle\psi\rangle$ is considered as an $\mathbb{R}$-subspace of $C^\infty(F)$. Notice that $\mathscr{K}_\varphi^\psi$ is not a real vector space unless ψ is identically zero.

In Section 4, we study KVFs of SSS-Ts. Firstly, we show necessary and sufficient conditions for a vector field of the form $h\partial_t + V$ to be a conformal Killing (see Proposition 4.2).

Then adapting the techniques of Sánchez in [2] to SSS-Ts, we give our first main result, namely.

Theorem 2.1. *Let* $f \in C^\infty_{>0}(F)$ *and* $I_f \times F := (I \times F, g := -f^2 \mathrm{d}t^2 \oplus g_F)$ *the* f*-associated SSS-T. Then, given an arbitrary* $t_0 \in I$, *the set of KVFs on* $I_f \times F$ *is*

$$\psi h \partial_t + \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\, f^2 \mathrm{grad}_F \psi + \widehat{K} + \mathscr{K}^0_{\ln f}, \tag{2.1}$$

where $h \in C^\infty(I)$ *verifies*

$$-h'' = \nu h, \quad \nu \in \mathbb{R}; \tag{2.2}$$

$\psi \in C^\infty(F)$ *verifies*

$$f^2 \mathrm{grad}_F \psi \in \mathscr{K}^{\nu\psi}_{\ln f} \neq \varnothing \tag{2.3}$$

and

$$\widehat{K} \in \mathscr{K}^{-h'(t_0)\psi}_{\ln f} \neq \varnothing, \tag{2.4}$$

where $\varnothing$ *is the empty set.*

If $\nu \neq 0$, *then* $-\frac{h'(t_0)}{\nu} f^2 \mathrm{grad}_F \psi$ *may be taken as* $\widehat{K}$ *and* (2.1) *takes the form*

$$\psi h \partial_t - \frac{h'}{\nu} f^2 \mathrm{grad}_F \psi + \mathscr{K}^0_{\ln f}. \tag{2.5}$$

We remark here the central role of the problem (2.3) in Theorem 2.1. Our approach essentially reduces (2.3) to the study of a parametric overdetermined system of partial differential equations (involving the Hessian) on the Riemannian part (F, g_F).