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Duality and cohomology in M-theory with boundary☆

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ABSTRACT

We consider geometric and analytical aspects of M-theory on a manifold with boundary Y^{11}. The partition function of the C-field requires summing over harmonic forms. When Y^{11} is closed, Hodge theory gives a unique harmonic form in each de Rham cohomology class, while in the presence of a boundary the Hodge–Morrey–Friedrichs decomposition should be used. This leads us to study the boundary conditions for the C-field. The dynamics and the presence of the dual to the C-field gives rise to a mixing of boundary conditions with one being Dirichlet and the other being Neumann. We describe the mixing between the corresponding absolute and relative cohomology classes via Poincaré duality angles, which we also illustrate for the M5-brane as a tubular neighborhood. Several global aspects are then considered. We provide a systematic study of the extension of the E_8 bundle and characterize obstructions. Considering Y^{11} as a fiber bundle, we describe how the phase looks like on the base, hence providing dimensional reduction in the boundary case via the adiabatic limit of the eta invariant. The general use of the index theorem leads to a new effect given by a gravitational Chern–Simons term CS_{11} on Y^{11} whose restriction to the boundary would be a generalized WZW model. This suggests that holographic models of M-theory can be viewed as a sector within this index-theoretic approach.

1. Introduction

M-theory has proven to be a rich theory both in terms of modeling physical phenomena and in terms of mathematical structures. Physical and mathematical insight could be gained by studying various aspects of this theory. In this paper, we study geometric and analytical aspects of M-theory on a manifold with a boundary, building on [1], mainly by emphasizing importance of boundary conditions and their effect on the corresponding bundles, fields, actions and partition functions. The main ingredient we use for the kinetic term is harmonic forms. In the presence of a boundary, the Hodge decomposition theorem has to be modified and new effects appear, depending on boundary conditions. For the phase we use index theory on manifolds with a boundary, departing from [1,2] by the use of the adiabatic limit of the eta-invariants.

We take M-theory on an eleven-dimensional Spin manifold with boundary Y^{11} equipped with a Riemannian metric g_Y. The main fields we consider are the C-field C_3 with its field strength G_4 as well the dual field G_7, the eleven-dimensional Hodge dual to G_4 at the level of differential forms. The C-field has a classical harmonic part (see e.g. [3]), which is characterized in [4] in the extension to the Spin bundle. The Bianchi identity and equation of motion for the C-field in M-theory, which follow from those of eleven-dimensional supergravity [5], are

$$dG_4 = 0, \qquad \frac{1}{\ell_p^3} d * G_4 = \frac{1}{2} G_4 \wedge G_4 - I_8, \tag{1.1}$$

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where I_8 is the one-loop polynomial, $*$ is the Hodge duality operation in eleven dimensions, and ℓ_p is the scale in the theory called the Planck 'constant'. A formulation in terms of G_4 and G_7 is given in [6]. The presence of $d*G_4 = dG_7$ suggests looking also at a degree eight field G_8 (this is called Θ in [1]). The two fields G_4 and G_8 can be treated in a unified way [7–9,4].

Harmonic C-field. The classical (or low energy) limit is obtained by taking $\ell_p \to 0$ and is dominated by the metric-dependent terms. In this long distance approximation of M-theory, one keeps only the harmonic modes of the C-field [3,4]. Let $\Delta_g^3 : \left(\Omega^3(Y^{11}), g\right) \longrightarrow \left(\Omega^3(Y^{11}), g\right)$ be the Hodge Laplacian on 3-forms on the base Y^{11} with respect to the metric g given by $\Delta_g^3 = d\,d^* + d^* d$, where d^* is the adjoint operator to the de Rham differential operator d. Assuming $[G_4] = 0$ in $H^4(Y^{11}; \mathbb{R})$ so that $G_4 = dC_3$ then in the Lorentz gauge, $d^* C_3 = 0$, we have [4] $\Delta_g^3 C_3 = *j_e$, where j_e is the electric current associated with the membrane given by $j_e = \ell_p^3 \left(\frac{1}{2} G_4 \wedge G_4 - I_8\right)$. Thus, C_3 is harmonic if $\ell_p \to 0$ and/or there are no membranes. The space of harmonic 3-forms on Y^{11} is $\mathcal{H}_g^3(Y^{11}) := \ker \Delta_g^3 \subset \Omega^3(Y^{11})$. Harmonic forms are very important in compactification, where the fields are expanded in a harmonic basis. For instance, if α^i is basis for the space $\mathcal{H}^3(Y^{11})$ of harmonic 3-forms on Y^{11}, then the C-field can be expanded as $C_3 = \sum_i C_3^i \alpha^i$. There are natural choices for internal manifolds for compactifications with fluxes leading to supersymmetric theories in lower dimensions (see [10] and the references therein). A seven-dimensional manifold M with a 3-form φ is a G_2 manifold if $d\varphi = d^*\varphi = 0$, that is if φ is harmonic. An eight-manifold with a self-dual four-form $\phi = *\phi$ is called a torsion-free Spin(7) manifold if $d\phi = 0$.

The C-field in the presence of a boundary. When Y^{11} has a boundary we no longer assume that there is a bounding twelve-manifold Z^{12}. The topological sectors of the C-field are labeled by extensions $\tilde{a}$ of the degree four characteristic class of the C-field on $M^{10} = \partial Y^{11}$. In addition to summing over torsion, there will be an integral over a certain space of harmonic fields. Consider the inclusion $i : M^{10} \hookrightarrow Y^{11}$, which induces the pullback on cohomology $i^* : H^4(Y^{11}; \mathbb{Z}) \to H^4(M^{10}; \mathbb{Z})$. In [1], the sum over the topological sectors in the wavefunction is restricted to $\ker i^* \subset H^4(Y^{11}; \mathbb{Z})$, which is equivalent to a sum over $H^4(Y^{11}, M^{10}; \mathbb{Z})/\delta H^3(M^{10}; \mathbb{Z})$, where δ is the connecting homomorphism, and the integration in the path integral would be over the compact space of harmonic forms $\mathcal{H}^3(Y^{11}, M^{10})/\mathcal{H}^3(Y^{11}, M^{10})_{\mathbb{Z}}$, where $\mathcal{H}^3(Y^{11}, M^{10}) := \ker i^*$ restricted to $\mathcal{H}^3(Y^{11})$. It is desirable to further characterize these, which is one of the goals of this paper. We formulate a boundary value problem which is solvable from general considerations in Section 2.2. We work with C_3 as well as its field strength so that both degree three and four cohomology are relevant.

Boundary conditions and duality. In the absence of the field dual to the C-field, the boundary conditions can be taken in a straightforward way. However, when this field is introduced, an interplay between Hodge duality and dynamics of the fields makes such obvious choices not possible. In particular, if the C-field is taken to satisfy the Dirichlet boundary condition, then its dual must satisfy the Neumann boundary condition, and vice versa. Thus, in this paper, we provide a systematic study of these matters. Naturally, then one might ask what replaces the duality in Y^{11} when one restricts to the boundary. We study analogs of the Hilbert transform introduced in [11] which effectively provides a description for such a duality and exchanges Dirichlet and Neumann forms. In addition, we will consider generalization to include the dual fields in Section 2.1.

Cohomology in the presence of boundary. An arbitrary de Rham cohomology class of an oriented compact Riemannian manifold can be represented by a unique harmonic form, i.e. the natural map $\mathcal{H}^k(M) \to H^k_{dR}(M)$ is an isomorphism. This means that every cohomology class contains exactly one harmonic form. When Y^{11} is closed then, from the Hodge decomposition theorem, the fourth cohomology group with real coefficients $H^4(Y^{11}; \mathbb{R})$ is isomorphic to the space of closed and coclosed differential 4-forms on Y^{11}. Thus, the space of these harmonic forms provides a concrete realization of the cohomology group $H^4(Y^{11}; \mathbb{R})$ inside the space $\Omega^4(Y^{11})$ of all 4-forms on Y^{11}. The Laplacian Δ on p-forms on a closed Y^{11} is self-adjoint. However, in the presence of a boundary this is no longer the case, and in fact Δ is surjective [12]. In this case, we use (in Section 2.2) the Hodge–Morrey–Friedrich (HMF) decomposition theorem [13,14], which gives us a concrete realization of the absolute cohomology $H^4(Y^{11}; \mathbb{R})$ and the relative cohomology $H^4(Y^{11}, \partial Y^{11}; \mathbb{R})$ inside the space of all harmonic 4-forms on Y^{11}. The two spaces, surprisingly, intersect only at zero (see [15]) $\mathrm{Harm}^4(Y^{11}, \partial Y^{11}; \mathbb{R}) \cap \mathrm{Harm}^4(Y^{11}; \mathbb{R}) = \{0\}$. In addition, the boundary subspace of each is orthogonal to all of the other. Each of $H^4(Y^{11}; \mathbb{R})$ and $H^4(Y^{11}, \partial Y^{11}; \mathbb{R})$ has a portion consisting of those cohomology classes coming from the boundary ∂Y^{11} and another portion of those coming from the interior part of Y^{11}. The principal angles between the interior subspaces of the concrete realizations of $H^4(Y^{11}; \mathbb{R})$ and $H^4(Y^{11}, \partial Y^{11}; \mathbb{R})$ are called *Poincaré duality angles* and are characterized, via a refinement of the Hodge–Morrey–Friedrich decomposition, in [16] and also [15]. Poincaré duality angles measure how near a manifold with boundary is to being closed. We will be interested in orthogonal decomposition and not just direct sum decomposition (kinetic terms etc.). Similar discussion is provided for other fields, namely C_3, G_7 and G_8. We highlight new effects on the fields due to such phenomena. All of this is discussed in Section 2.3. Considering the M5-brane as a tubular neighborhood in the ambient spacetime we illustrate, in Section 2.4, the dependence of kinetic terms on distance scales.

E_8 gauge theory for $\partial Y^{11} \neq \emptyset$. The phase of the (non-gravitational) partition function can be studied using E_8 gauge theory [17]. For each characteristic class $a \in H^4(Y^{11}; \mathbb{Z})$ of an E_8 bundle over Y^{11}, there is a harmonic four-form G_4^a of the appropriate topological class. The kinetic energy $|G_4^a|^2 = \int_{Y^{11}} G_4^a \wedge *G_4^a$ vanishes if and only if G_4^a is torsion. The partition function involves evaluating the sum $\sum_{a \in H^4(Y^{11};\mathbb{Z})} (-1)^{f(a)} \exp\left(-|G_4^a|^2\right)$, where $f(a)$ is a quadratic refinement of a bilinear form related to a [17]. In dealing with boundaries, one has to impose boundary conditions on the C-field within the E_8 model. In [1], the conditions $i^*(C) = 0$ is chosen, where $\check{C} = (A, C)$, A is a connection on an E_8 bundle. This restricts to the C-fields $\mathrm{E}_P(Y^{11}, M^{10}) := \{(A, C) \in \mathrm{E}_P(Y^{11}) \mid i^*(C) = 0\}$. The boundary condition breaks the topological gauge symmetry $\mathcal{G}$, which