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The solution of the global relation for the derivative nonlinear Schrödinger equation on the half-line

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ABSTRACT

We consider initial-boundary value problems for the derivative nonlinear Schrödinger (DNLS) equation on the half-line $x > 0$. In a previous work, we showed that the solution $q(x, t)$ can be expressed in terms of the solution of a Riemann–Hilbert problem with jump condition specified by the initial and boundary values of $q(x, t)$. However, for a well-posed problem, only part of the boundary values can be prescribed; the remaining boundary data cannot be independently specified, but are determined by the so-called global relation. In general, an effective solution of the problem therefore requires solving the global relation. Here, we present the solution of the global relation in terms of the solution of a system of nonlinear integral equations. This also provides a construction of the Dirichlet-to-Neumann map for the DNLS equation on the half-line.

1. Introduction

The derivative nonlinear Schrödinger (DNLS) equation

$$iq_t + q_{xx} = i\left(|q|^2 q\right)_x, \tag{1.1}$$

where $q(x, t)$ is a complex-valued function, arises in the study of wave propagation in optical fibers [1] and in plasma physics [2] (see [3] for further references). It is an integrable equation and the initial value problem on the line can be analyzed by means of the Inverse Scattering Transform (IST) as demonstrated by Kaup and Newell [4]. In the last fifteen years, a generalization of the IST to initial-boundary value (IBV) problems developed by Fokas and his collaborators [5–7], has made it possible to analyze equations such as (1.1) on domains involving a boundary. Several of the most well-known integrable PDEs (such as the KdV, modified KdV, nonlinear Schrödinger (NLS), sine-Gordon, and Ernst equations) have been investigated using this approach, see e.g. [8–15].

Recently, the Fokas method was implemented to the DNLS equation (1.1) posed on the half-line $x > 0$ [16].[1] Provided that the solution exists, it was shown in [16] that the solution $q(x, t)$ can be recovered from the initial and boundary data via the solution of a 2×2-matrix Riemann–Hilbert (RH) problem. The jump matrix for this RH problem is given explicitly in terms of four spectral functions $a(k)$, $b(k)$, $A(k)$, and $B(k)$, where $k \in \mathbb{C}$ is the spectral parameter of the associated Lax pair. The functions $a(k)$ and $b(k)$ are defined in terms of the initial data $q_0(x) = q(x, 0)$ via a system of linear Volterra integral equations. The functions $A(k)$ and $B(k)$ are defined in terms of the boundary data $g_0(t) = q(0, t)$ and $g_1(t) = q_x(0, t)$ also via a system of linear Volterra integral equations. However, for a well-posed problem, only one of the functions g_0 and g_1 (or a combination of these two functions) can be specified; the remaining boundary data cannot be independently specified, but are determined by the so-called global relation. Thus, before the functions $A(k)$ and $B(k)$ can be constructed from the above linear integral equations, the global relation must first be used to eliminate the unknown boundary data.

The analysis of the global relation can take place in two different domains: in the physical domain or in the spectral domain. Although these two domains are related by a transform, each viewpoint has its own advantages.

In the first part of this paper, we analyze the global relation in the spectral domain. We will show that $A(k)$ and $B(k)$ can be determined via the solution of a system of nonlinear integral equations formulated explicitly in terms of the initial data and the known boundary values.

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[1] Physically, this type of initial-boundary value problem arises naturally. For example, assuming that we can create or measure the waves at some fixed point $x = 0$ in space, we arrive at an initial-boundary value problem on the half-line with initial data given by the initial wave profile for $x > 0$ and boundary data provided by the measurements at $x = 0$.

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In the second part of the paper, we analyze the global relation in the physical domain. We use a Gelfand–Levitan–Marchenko (GLM) representation to derive an expression for the generalized Dirichlet-to-Neumann map (i.e. the map which determines the unknown boundary values from the known ones). Once the unknown boundary values have been determined, $A(k)$ and $B(k)$ can be constructed from the linear integral equations mentioned earlier. This representation has the numerical advantage that the system of integral equations is defined on a bounded domain.

In the case of the NLS equation, a construction of the Dirichlet-to-Neumann map was presented in [17,18]. The Dirichlet-to-Neumann map for the sine-Gordon equation as well as the two versions of mKdV were analyzed in [18]. The analysis of the KdV equation presents some novel difficulties which were finally overcome in [19]. Although our approach is inspired by the preceding references, the analysis here presents a number of novelties: (a) The solution of the global relation presented in Section 3 takes place entirely in the spectral domain. This is in contrast to the approach of [18], in which it is necessary to derive a GLM representation for an appropriate eigenfunction of the Lax pair before an analogous result can be obtained. The introduction of a GLM representation amounts to passing from the spectral to the physical plane. Here we remain in the spectral plane throughout the derivation, which simplifies the arguments both on a practical and on a conceptual level. The observation that it is possible to solve the global relation directly in the spectral plane was first made for the NLS equation in [20]. (b) In the special, but important, case of vanishing initial data $q_0 = 0$, we will show that a further analysis of the global relation can be used to simplify the integral equations derived in the physical plane considerably. In fact, the number of equations reduces from five to two in this case. The case of vanishing initial data is of special interest in applications, where it can sometimes be assumed that the physical field under consideration is initially at rest before exteriorly created waves enter the domain. This is also the relevant initial condition in the application of the numerical methods based on the so-called "exact nonreflecting boundary conditions" [21]. A similar simplification for the NLS equation was presented in [20]. (c) The Lax pair for Eq. (1.1) contains terms of higher order in the spectral parameter k than what is the case for the examples above. However, by utilizing additional symmetries, calculations can still be kept at a reasonable length. (d) As noted in [16], the definition of eigenfunctions with the appropriate asymptotics of the Lax pair of (1.1) naturally leads to the introduction of a certain closed two-form Δ. In order to close the system of integral equations which characterizes the solution of the global relation, the system must be supplemented by an additional (somewhat complicated) equation for the second component of Δ.

The paper is organized as follows: In Section 2, we recall the Lax pair formulation and the global relation associated with (1.1). In Sections 3 and 5, the global relation is analyzed in the spectral and physical domains, respectively. In Section 4, we derive a GLM representation for an appropriate eigenfunction of the Lax pair.

2. A Lax pair and the global relation

Eq. (1.1) admits the Lax pair [4]

$$\begin{cases} \Psi_x + ik^2[\sigma_3, \Psi] = U_1\Psi, \\ \Psi_t + 2ik^4[\sigma_3, \Psi] = U_2\Psi, \end{cases} \tag{2.1}$$

where $\sigma_3 = \operatorname{diag}(1, -1)$, $k \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a spectral parameter, $\Psi(x, t, k)$ is a 2×2-matrix valued eigenfunction, and the 2×2-matrix valued functions $Q(x, t)$ and $\{U_j(x, t, k)\}_1^2$ are defined by

$$Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \qquad U_1 = kQ, \qquad U_2 = 2k^3Q - ik^2Q^2\sigma_3 - ikQ_x\sigma_3 + kQ^3.$$

Following [16], we define a transformed eigenfunction $\mu(x, t, k)$ by

$$\Psi(x, t, k) = e^{i\int_{(0,0)}^{(x,t)} \Delta\sigma_3}\mu(x, t, k)e^{i\int_{(\infty,0)}^{(0,0)} \Delta\sigma_3}, \tag{2.2}$$

where $\Delta = \Delta_1 \mathrm{d}x + \Delta_2 \mathrm{d}t$ is the closed real-valued one-form

$$\Delta(x, t) = \frac{1}{2}|q|^2 \mathrm{d}x + \left(\frac{3}{4}|q|^4 - \frac{i}{2}(\bar{q}_x q - \bar{q}q_x)\right) \mathrm{d}t. \tag{2.3}$$

We introduce $\mathcal{Q}(x, t)$ and $\mathcal{Q}_1(x, t)$ by

$$\mathcal{Q} = e^{-i\int_{(0,0)}^{(x,t)} \Delta\hat{\sigma}_3}Q, \qquad \mathcal{Q}_1 = e^{-i\int_{(0,0)}^{(x,t)} \Delta\hat{\sigma}_3}Q_x,$$

where $\hat{\sigma}_3$ acts on a 2×2 matrix A by $\hat{\sigma}_3 A = [\sigma_3, A]$, i.e. $e^{\hat{\sigma}_3}A = e^{\sigma_3}Ae^{-\sigma_3}$. The function μ satisfies the Lax pair

$$\begin{cases} \mu_x + ik^2[\sigma_3, \mu] = V_1\mu, \\ \mu_t + 2ik^4[\sigma_3, \mu] = V_2\mu, \end{cases} \tag{2.4}$$

where the 2×2-matrix valued functions $\{V_j(x, t, k)\}_1^2$ are given by

$$V_1 = k\mathcal{Q} - i\Delta_1\sigma_3, \qquad V_2 = 2k^3\mathcal{Q} - ik^2\mathcal{Q}^2\sigma_3 - ik\mathcal{Q}_1\sigma_3 + k\mathcal{Q}^3 - i\Delta_2\sigma_3. \tag{2.5}$$

Let $T > 0$ be some given final time; we will assume that $T < \infty$. Assume that $q(x, t)$ is a solution of (1.1) with sufficient smoothness in the domain $\{0 < x < \infty, 0 < t < T\}$ and with sufficient decay as $x \to \infty$. We introduce three solutions $\{\mu_j(x, t, k)\}_1^3$ of (2.4) as the solutions of the following linear Volterra integral equations:

$$\mu_j(x, t, k) = I + \int_{(x_j,t_j)}^{(x,t)} e^{i[k^2(x'-x)+2k^4(t'-t)]\hat{\sigma}_3}\left((V_1\mu_j)(x', t', k)\mathrm{d}x' + (V_2\mu_j)(x', t', k)\mathrm{d}t'\right), \tag{2.6}$$