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Bouncing dynamics of a spring

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HIGHLIGHTS

- We propose a model for the dynamics of a deformable bouncing object.
- We obtain the equation of motion and simulate numerically the bouncing dynamics.
- Bifurcation diagrams illustrated the influence of the deformations in the dynamics.
- The bouncing threshold curve is analyzed theoretically and numerically.

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ABSTRACT

We consider the dynamics of a deformable object bouncing on an oscillating plate and we propose to model its deformations. For this purpose, we use a spring linked to a damper. Elastic properties and viscous effects are taken into account. From the bouncing spring equations of motion, we emphasize the relevant parameters of the dynamics. We discuss the range of parameters in which elastic deformations do not influence the bouncing dynamics of this object and compare this behavior with the bouncing ball dynamics. By calculating the spring bouncing threshold, we evidence the effect of resonance and prove that elastic properties can make the bounce easier. This effect is for example encountered in the case of bouncing droplets. We also consider bifurcation diagrams in order to describe the consequences of a dependence on the frequency. Finally, hysteresis in the dynamics is presented.

1. Introduction

The bouncing ball problem is a straightforward experiment which illustrates complex deterministic dynamical systems [1]. This experiment considers a solid ball bouncing onto a rigid plate that vertically oscillates. The bouncing is possible according to two mechanisms. The first one is the impulsion provided by the plate. The driving parameter is the acceleration Γ which measures the maximal acceleration of the plate in g units. The second one is the restitution in kinetic energy at each impact. The restitution coefficient ϵ is defined as the ratio between the ball velocity after and before each impact. Despite the apparent simplicity of this model, several behaviors can be observed. For example, depending on both parameters Γ and ϵ, the bouncing ball shows bifurcations in its periodic bouncing modes, i.e. a change of periodicity in the way the ball bounces. The ball may even bounce chaotically. This system has been largely studied through past decades as reported in the articles of Luck et al. [2], Juo et al. [3] or more recently in the article of Gilet et al. [4]. Indeed, bifurcations in the bounce have been detailed and the stability of the dynamics has also been studied. Furthermore, periodicity and chaotic behaviors have been investigated. One also reports numerous applications of the bouncing ball model in various fields such as granular media [5], nanotechnology [6], neuro-sciences [7], gambling [8] or in the bouncing droplets dynamics [9].

Numerous variations have also been investigated such as the quantum version of the bouncing ball [10] or the bouncing ball on an elastic membrane [11]. This last article describes the effects of the deformations of the oscillating plate. The system has an analogue in fluid mechanics where a droplet can bounce onto an oscillating soap film [12,13]. Additional degrees of freedom have also been included in the dynamics. One can report the experiments on bouncing dimers [14] and bouncing trimers [15]. In those papers, two or three metallic beads are linked together and the formed rigid object possesses new degrees of freedom compared with the bouncing ball. As a consequence, horizontal motions and rotations of the compound object are observed.

As far as we know, a model considering the deformations of a bouncing object and their effects in the motion is still lacking. Such effects have already been observed in bouncing droplets as reported in the articles [12,16–18]. Indeed, drop deformations store potential energy thanks to surface tension effects. Thus, we propose to investigate the role of deformations by using a spring and a

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damper and studying the bounce of the compound object on a rigid and oscillating plate. In our model, the spring, at impact, stores potential energy that must be taken into account in the dynamics. Simultaneously, the damper dissipates this energy during the oscillations of the spring. Thus, those components aim to model the elastic and viscous effects of a bouncing object. Unlike the bouncing ball, this bouncing spring is not a point particle and its length varies during its motion. Furthermore, since the spring is characterized by its natural frequency, its dynamics may change with the plate frequency. In particular, the bouncing spring may resonate. This model gives us a handy way to understand the dynamics of the bouncing droplets, as explained above.

The bouncing ball and bouncing spring models will be described and compared in Section 2. In Section 3, we will determine the set of parameters that confers the bouncing spring the properties of a bouncing ball, i.e. the range of parameters in which the elastic properties of the spring are negligible. Afterwards, the elastic properties will be considered and studied through the bouncing threshold of the spring. Resonance effects will be highlighted. Some bifurcation diagrams will be provided in order to illustrate the effects of deformations. Finally, hysteresis will be described and analyzed before summarizing the work.

2. Models

2.1. Bouncing ball

The bouncing ball dynamics is the starting point of our work. The bouncing ball model is illustrated on Fig. 1(left) and considers a mass which bounces on a rigid plate oscillating sinusoidally with an angular frequency ω and an amplitude A. The plate vertical position z_p is described by

$$z_p(t) = A\cos(\omega t). \tag{1}$$

The ball is only submitted to gravity g and to its successive interactions with the plate. Thus, in the laboratory frame, between two successive impacts, the acceleration of the ball is given by

$$\frac{d^2(z_b - z_p)}{dt^2} = -g + A\omega^2\cos(\omega t), \tag{2}$$

where $z_b(t)$ is the height of the ball. Defining the dimensionless time $\phi = \omega t$, the dimensionless height of the ball $\alpha_b = z_b/A$ and the reduced acceleration of the plate $\Gamma = A\omega^2/g$, one obtains

$$\frac{d^2(\alpha_b - \alpha_p)}{d\phi^2} = -\frac{1}{\Gamma} + \cos(\phi). \tag{3}$$

One needs another equation to complete the description of the bounces of the ball. Impacts are described as follows: the speed of the ball after impact $(+)$ is linked to the speed before impact $(-)$ by

$$\left.\frac{d(\alpha_b - \alpha_p)}{d\phi}\right|_+ = \epsilon \left.\frac{d(\alpha_b - \alpha_p)}{d\phi}\right|_-, \tag{4}$$

where ϵ is the coefficient of restitution. Please, note that the impact is seen as instantaneous. From Eqs. (3) and (4), one observes that the dynamics is driven by only two parameters: Γ and ϵ. A more complete description of the bouncing ball dynamics is provided by Gilet et al. [4] for completely inelastic impacts.

Both parameters Γ and ϵ characterize the mechanisms that enable the ball to bounce. The parameter Γ is linked to the inertial effects in the system. When $\Gamma > 1$ the ball takes-off only because of the motion of the plate. The parameter ϵ characterizes the impact dissipation. A non-zero value of this parameter allows the ball to leave the plate after an impact.

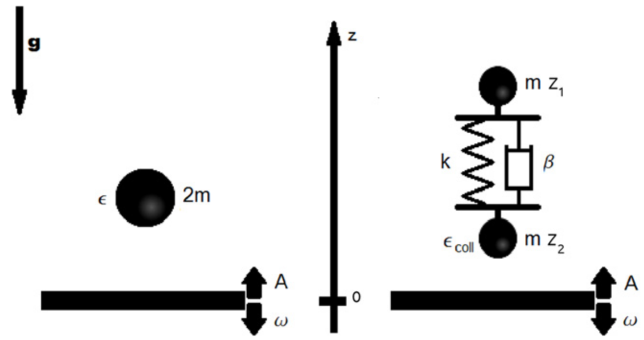

Fig. 1. Schematic representation of both models: bouncing ball (left) and bouncing spring (right). On the left, the bouncing ball model considers a point particle bouncing on a solid plate. This plate oscillates with an amplitude A at angular frequency ω. The impact elasticity is tuned through a parameter ϵ. On the right, the bouncing spring model is represented. It consists of two identical masses m linked with a spring of stiffness k and a damper of viscosity β. The whole system bounces onto a solid surface oscillating sinusoidally with an amplitude A and an angular frequency ω. The contact between the bottom mass of the bouncing spring and the oscillating plate is tuned through the coefficient of restitution ϵ_{coll}.

2.2. Bouncing spring

We consider that the bouncing object is vertically deformable. In order to understand this effect, we propose to model the object by two identical masses m linked by a spring of stiffness k and length at rest L. The vertical positions are respectively z_1 for the upper mass and z_2 for the lower mass. Viscous effects into the bouncing spring are modeled by a dash-pot and tuned through the dissipation parameter β. A schematic representation of the bouncing spring is provided on Fig. 1(right). Just like the bouncing ball, the compound object bounces onto a rigid surface which oscillates according to Eq. (1). One can figure out that this system is characterized by a natural frequency and thus, a deep dependence of the oscillation frequency, and resonance are expected. The motion of both masses is described by the following set of Newton's equations in the laboratory frame

$$\begin{cases} m\dfrac{d^2z_1}{dt^2} + \beta\left(\dfrac{dz_1}{dt} - \dfrac{dz_2}{dt}\right) + k(z_1 - z_2 - L) + mg = 0, \\ m\dfrac{d^2z_2}{dt^2} - \beta\left(\dfrac{dz_1}{dt} - \dfrac{dz_2}{dt}\right) - k(z_1 - z_2 - L) + mg = N_2(t) \end{cases} \tag{5}$$

where $N_2(t)$ is the normal reaction of the surface. The shape of the normal reaction is unknown since it depends on the spring motion, but it can be evaluated through the molecular dynamics algorithm. This technique, described here-below, needs to use a restitution coefficient ϵ_{coll} in order to determine the amount of energy lost by the lower mass during impact. One has to understand that this coefficient is strictly identical to the one introduced during the study of the bouncing ball. Note that in the spring case, the contact with the plate is finite in time and not instantaneous like a bouncing ball. Defining the natural angular frequency of the spring as $\omega_0 = \sqrt{k/m}$ and the dissipation coefficient as $\xi = \beta/2m\omega_0$, those equations can be rewritten as

$$\begin{cases} \dfrac{d^2z_1}{dt^2} + 2\xi\omega_0\left(\dfrac{dz_1}{dt} - \dfrac{dz_2}{dt}\right) + \omega_0^2(z_1 - z_2 - L) + g = 0, \\ \dfrac{d^2z_2}{dt^2} - 2\xi\omega_0\left(\dfrac{dz_1}{dt} - \dfrac{dz_2}{dt}\right) - \omega_0^2(z_1 - z_2 - L) + g \\ \quad = N_2(t)/m. \end{cases} \tag{6}$$

If one introduces the dimensionless length $\alpha = z/A$, the dimensionless time $\phi = \omega t$, the dimensionless frequency $\Omega = \omega/\omega_0$ and the dimensionless acceleration $\Gamma = A\omega^2/g$, one can write

$$\alpha_p(\phi) = \cos\phi \tag{7}$$