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The ADHM variety and perverse coherent sheaves

Marcos Jardim [a,*], Renato Vidal Martins [b]

[a] IMECC - UNICAMP, Departamento de Matemática, Caixa Postal 6065, 13083-970 Campinas-SP, Brazil

[b] ICEx - UFMG, Departamento de Matemática, Av. Antônio Carlos 6627, 30123-970 Belo Horizonte MG, Brazil

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ABSTRACT

We study the full set of solutions to the ADHM equation as an affine algebraic set, the ADHM variety. We determine a filtration of the ADHM variety into subvarieties according to the dimension of the stabilizing subspace. We compute dimension, and analyze singularity and reducibility of all of these varieties. We also establish a connection between arbitrary solutions of the ADHM equation and coherent perverse sheaves on $\mathbb{P}^2$ in the sense of Kashiwara.

1. Introduction

Atiyah et al. showed in [1], using Ward correspondence and algebro-geometric techniques (monads) introduced by Horrocks, that all self-dual connections on euclidean 4-dimensional space (a.k.a. instantons) have a unique description in terms of linear algebra.

A few years later in [2], Donaldson restated the ADHM description in terms of the following data. Let V and W be complex vector spaces, with dimensions c and r, respectively. Let $A, B \in \mathrm{Hom}(V, V)$, let $I \in \mathrm{Hom}(W, V)$ and let $J \in \mathrm{Hom}(V, W)$. Consider the following equations:

$$[A, B] + IJ = 0 \tag{1}$$

$$[A, A^\dagger] + [B, B^\dagger] + II^\dagger - J^\dagger J = 0. \tag{2}$$

The group $GL(c)$ acts on the set of solutions of the first equation, sending a datum (A, B, I, J) to $(gAg^{-1}, gBg^{-1}, gI, Jg^{-1})$, where $g \in GL(c)$, while the unitary group $U(c)$ preserves the second equation. Among other things, Donaldson showed that the *regular* (Definition 2.1) solutions to Eq. (1) modulo the action of $GL(c)$ parametrize the moduli space of holomorphic bundles on $\mathbb{CP}^2$ that are framed at a line, the so-called line at infinity. The key observation is that the set of regular solutions of the first equation modulo $GL(c)$ is isomorphic to the set of regular solutions of both equations modulo $U(c)$, which in turn is identified with the moduli space of framed instantons on $\mathbb{R}^4$.

More recently, Nakajima showed that relaxing the regularity condition to a weaker stability condition one obtains the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$, see [3]. In particular, Nakajima also obtained a linear algebraic description of the Hilbert scheme of points on $\mathbb{C}^2$ and showed that it admits a hyper-Kähler structure.

Generalizations of the ADHM construction have given rise to a wide variety of important results in many different areas of mathematics and mathematical physics. For instance, Kronheimer and Nakajima constructed instantons on the so-called ALE manifolds [4]. Nakajima then proceeded to further generalize the construction, and studied quiver varieties

* Corresponding author.
E-mail addresses: jardim@ime.unicamp.br (M. Jardim), renato@mat.ufmg.br (R.V. Martins).

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and representations of Kac–Moody Lie algebras [5]. More recently, the ADHM construction of instantons was adapted to construct analogues of Yang–Mills instantons in noncommutative geometry [6] and supergeometry [7], while the ADHM construction of holomorphic bundles on $\mathbb{CP}^2$ was generalized to the construction of instanton bundles on $\mathbb{CP}^n$ [8,9].

We start this paper by studying the full set of solutions of (1), called the *ADHM equation*, as an affine algebraic variety $\mathcal{V}(r,c)$, called the *ADHM variety*. It is known that $\mathcal{V}(r,c)$ is a complete intersection affine variety of dimension $2rc+c^2$ which is irreducible if $r \geq 2$ [10, Theorem 1.2], and, if $r = 1$, it has $c+1$ equidimensional irreducible components [11, Theorem 1.1].

Our first concern are the points $X \in \mathcal{V}(r,c)$ which are not *stable*. In order to study them, we define the *stabilizing subspace* Σ_X as the subspace of V to which the restriction of X is stable (c.f. Definition 2.5). We break down the ADHM variety into disjoint subsets $\mathcal{V}(r,c)^{(s)} := \{X \mid \dim \Sigma_X = s\}$; in this sense, the set of stable points $\mathcal{V}(r,c)^{\mathrm{st}}$ now corresponds to $\mathcal{V}(r,c)^{(c)}$. We also consider the closed subvarieties $\mathcal{V}(r,c)^{[s]} := \{X \mid \dim \Sigma_X \leq s\}$ and analyze them as well.

Next, it is important for us to put the subject within a categorical framework. We pass from "points of the ADHM variety $\mathcal{V}$" to "objects in the category $\mathcal{A}$ of representations of the *ADHM quiver*". Such objects are of the form $\mathbf{R} = (V, W, X)$ where V, W and X are as above. For each $\mathbf{R} \in \mathcal{A}$ we define $\mathbf{S_R} := (\Sigma_X, W, X|_{\Sigma_X})$ as its *stable restriction* and $\mathbf{Z_R} := (V/\Sigma_X, \{0\}, (A', B', 0, 0))$ as its *quotient representation*, with A', B' commuting operators in V/Σ_X, see (7).

From every $\mathbf{R} \in \mathcal{A}$ one constructs a complex $E^\bullet_{\mathbf{R}} \in \mathrm{Kom}(\mathbb{P}^2)$ (Definition 5.1), called the *ADHM complex* associated to $\mathbf{R}$. Nakajima has shown in [3, Chapter 2], based on previous constructions due to Barth and Donaldson [2], that if $\mathbf{R}$ is stable, then $E := \mathcal{H}^0(E^\bullet_{\mathbf{R}})$ is the only nontrivial cohomology sheaf of the ADHM complex; moreover, E is a torsion free sheaf such that $E|_\ell \simeq W \otimes \mathcal{O}_\ell$ for a fixed line $\ell \subset \mathbb{P}^2$. Conversely, given any torsion free sheaf E on $\mathbb{P}^2$ whose restriction to ℓ is trivial, then there is a stable solution of the ADHM equation such that E is isomorphic to the cohomology sheaf of the corresponding ADHM complex. Besides, $\mathcal{H}^0(E^\bullet_{\mathbf{R}})$ is locally free if and only if $\mathbf{R}$ is regular.

Following Drinfeld's ideas, Braverman et al. showed, generalizing [3, Chapter 2], that arbitrary solutions of the ADHM equation correspond to what was defined as *perverse coherent sheaves* in [12, Section 5]; see also [13]. Here, we show that the complex associated to such solutions can also be regarded as perverse sheaves in the sense of Kashiwara's "family of supports" approach. More precisely, we introduce, following [14], a t-structure on $D^{\mathrm{b}}(\mathbb{P}^2)$; objects in the core of such a t-structure, denoted $\mathcal{P}$, are what we call *perverse coherent sheaves on* $\mathbb{P}^2$. We characterize objects $E^\bullet$ in $\mathcal{P}$, and define the *rank, charge* and *length* of such an $E^\bullet$ as, respectively, $\mathrm{rank}(\mathcal{H}^0(E^\bullet))$, $c_2(\mathcal{H}^0(E^\bullet))$ and $\mathrm{length}(\mathcal{H}^1(E^\bullet))$.

The results we obtain are summarized below.

Theorem. *The set $\mathcal{V}(r,c)^{(s)}$ is an irreducible, quasi-affine variety of dimension $2rc + c^2 - (r-1)(c-s)$ which is nonsingular iff either $s = c$ or $s = c-1$. The set $\mathcal{V}(r,c)^{[s]}$ is an affine variety which is irreducible if and only if $r \geq 2$. Moreover, for $0 \leq i \leq s$, the closures of $\mathcal{V}(1,c)^{(i)}$ in $\mathcal{V}(1,c)^{[s]}$ are precisely the irreducible components of $\mathcal{V}(1,c)^{[s]}$. If $\mathbf{R}$ is a representation of the ADHM associated to a point $X \in \mathcal{V}(r,c)^{(s)}$, then the associated ADHM complex $E^\bullet_{\mathbf{R}}$ is a perverse coherent sheaf on $\mathbb{P}^2$ of rank r, charge $c-s$ and length s such that*

(i) $\mathcal{H}^0(E^\bullet_{\mathbf{R}}) \simeq \mathcal{H}^0(E^\bullet_{\mathbf{S_R}})$;
(ii) $\mathcal{H}^1(E^\bullet_{\mathbf{R}}) \simeq \mathcal{H}^1(E^\bullet_{\mathbf{Z_R}})$;
(iii) $\mathcal{H}^0(E^\bullet_{\mathbf{R}})$ *is locally free if* $\mathbf{R}$ *is costable.*

In other words, the varieties $\mathcal{V}(r,c)^{(s)}$ parametrize perverse coherent sheaves on $\mathbb{P}^2$ of fixed rank, charge and length.

These statements are proved in Theorems 3.3 and 5.5. We also provide alternative proofs for the irreducibility of $\mathcal{V}(r,c)$ when $r \geq 2$ and for the counting of irreducible components of $\mathcal{V}(1,c)$, facts originally established in [10, Theorem 1.2] and [11, Theorem 1.1], respectively. Our characterization of $E^\bullet_{\mathbf{R}}$ as a perverse coherent sheaf is also different from what was previously considered in the literature (c.f. [12,13]).

2. The ADHM data

Let V and W be complex vector spaces, with dimensions c and r, respectively. The *ADHM data* is the set (or space) given by

$$\mathbf{B} = \mathbf{B}(r,c) := \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W).$$

An element (or point) of $\mathbf{B}$ is a *datum* $X = (A, B, I, J)$ with $A, B \in \mathrm{End}(V)$, $I \in \mathrm{Hom}(W,V)$ and $J \in \mathrm{Hom}(V,W)$.

Definition 2.1. A datum $X = (A, B, I, J) \in \mathbf{B}$ is said to be

(i) *stable* if there is no subspace $S \subsetneq V$ with $A(S), B(S), I(W) \subset S$;
(ii) *costable* if there is no subspace $0 \neq S \subset V$ with $A(S), B(S) \subset S \subset \ker J$;
(iii) *regular* if it is both stable and costable.

We call $\mathbf{B}^{\mathrm{st}}$, $\mathbf{B}^{\mathrm{cs}}$ and $\mathbf{B}^{\mathrm{reg}}$ the sets of stable, costable and regular data, respectively.