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Continuous symmetry reduction and return maps for high-dimensional flows

Evangelos Siminos *, Predrag Cvitanović

Center for Nonlinear Science, School of Physics, Georgia Institute of Technology, Atlanta, GA 30332-0430, United States

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ABSTRACT

We present two continuous symmetry reduction methods for reducing high-dimensional dissipative flows to local return maps. In the Hilbert polynomial basis approach, the equivariant dynamics is rewritten in terms of invariant coordinates. In the method of moving frames (or method of slices) the state space is sliced locally in such a way that each group orbit of symmetry-equivalent points is represented by a single point. In either approach, numerical computations can be performed in the original state space representation, and the solutions are then projected onto the symmetry-reduced state space. The two methods are illustrated by reduction of the complex Lorenz system, a five-dimensional dissipative flow with rotational symmetry. While the Hilbert polynomial basis approach appears unfeasible for high-dimensional flows, symmetry reduction by the method of moving frames offers hope.

1. Introduction

In his seminal paper, Lorenz [1] reduced the continuous time and discrete spatial symmetries of the three-dimensional Lorenz equations, resulting in a one-dimensional return map that yields deep insights [2] into the nature of chaos in this flow. For strongly contracting, low-dimensional flows, Gilmore et al. [3,4] systematized construction of such discrete time return maps, through the use of topological templates, Poincaré sections (to reduce the continuous time invariance) and invariant polynomial bases (to reduce the spatial symmetries). They showed that in the presence of spatial symmetries one has to 'quotient' the symmetry and replace the dynamics by a physically equivalent reduced, desymmetrized flow, in which each family of symmetry-related states is replaced by a single representative. This approach leads to symbolic dynamics and labeling of all periodic orbits up to a given topological period. Periodic orbit theory can then yield accurate estimates of long-time dynamical averages, such as Lyapunov exponents and escape rates [5].

In a series of papers, Cvitanović et al. [6–11] showed that effectively low-dimensional return maps can be constructed for high-dimensional (formally infinite-dimensional) flows described by dissipative partial differential equations (PDEs) such as the Kuramoto–Sivashinsky equation (KS). Such flows have state space topology vastly more complicated than the Lorenz flow, and collections of local Poincaré sections together with maps from a section to a section are required to capture all of the important asymptotic dynamics. These KS studies were facilitated by a restriction to the flow-invariant subspace of odd solutions, but at a price: elimination of the translational symmetry of the KS system and with it physically important phenomena, such as traveling waves. Traveling (or relative) unstable coherent solutions are ubiquitous and play a key role in the organization of turbulent hydrodynamic flows, as pointed out already in 1982 by Rand [12], and confirmed by both simulations and experimentation [13–18]. For KS [19,20], and even for a relatively low-dimensional flow such as the complex Lorenz equations [21,22] used as an example here, with the simplest possible continuous (rotational) spatial symmetry, the symmetry-induced drifts obscure the underlying hyperbolic dynamics.

The question that we address here is that of how one can construct suitable return maps for arbitrarily high-dimensional but strongly dissipative flows in the presence of continuous symmetries. Our exposition is based in part on Refs. [5,20,23]. The reader is referred to [24–28] for more depth and rigor than would be wise to wade into here.

In Section 2 we review the basic notions of symmetry in dynamics. Section 2.1 introduces the SO(2)-equivariant complex Lorenz equations (CLE), a five-dimensional set of ODEs that we use throughout the paper to illustrate the strengths and drawbacks of different symmetry reduction methods. In Section 3 we describe important classes of solutions and their symmetries: equilibria, relative equilibria, periodic and relative periodic orbits, and use them to motivate the need for symmetry reduction.

In Section 4 we describe the problem of *symmetry reduction*. The action of a symmetry group endows the state space with the structure of a union of group orbits, each group orbit an equivalence class. The goal of symmetry reduction is to replace each group orbit by a unique point in a lower-dimensional *reduced state space*. In Section 5 we briefly review the standard approach to spatial symmetry reduction, projection to a Hilbert basis, and explain why we

* Corresponding author.
E-mail address: siminos@gatech.edu (E. Siminos).

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find it impracticable. In Section 6 we review the method of moving frames, a direct and efficient method for computing symmetry-invariant bases that goes back to Cartan, and in Section 6.1 we apply the method to the complex Lorenz equations. The method maps all solutions to a slice, a submanifold of state space that plays a role for group orbits akin to the role that Poincaré sections play in reducing continuous time invariance. In contrast to the Hilbert basis approach, slices are local, with a generic trajectory within a slice bound to encounter singularities, and more than one slice might be needed to capture the flow globally. In Section 6.2 we show that a single local slice can suffice for the purpose of reducing the complex Lorenz equations flow to a return map. In Section 7 we recast the method of moving frames into the equivalent, differential method of slices, with time integration restricted to a slice fixed by a given state space point.

2. Symmetries of dynamical systems

Here we are interested in the role continuous symmetries play in dynamics. The methods that we develop are in principle applicable to translational and rotational symmetries of ODEs and PDEs, described by compact or non-compact Lie groups. We have in mind applications to PDEs such as Kuramoto–Sivashinsky and plane Couette flow which exhibit translational symmetries in either infinite or periodic domains. In the former case the group of symmetries is Euclidean and non-compact, and in the latter case it is orthogonal and compact. In numerical computations the periodic setting is usually considered and, through Fourier analysis, a translation is represented by the action of the one-parameter Lie SO(2) group on its linearly irreducible subspaces, the Fourier modes. Through truncation (for example spectral discretization), PDEs are transformed to high-dimensional, but finite-dimensional, systems of ODEs. The key concepts will thus be illustrated by a specific ODE example, the SO(2) group acting on a five-dimensional state space, linearly decomposable into a direct sum of irreducible subspaces of SO(2).

Consider a system of ODEs of the form

$$\dot{x} = v(x) \tag{1}$$

with v a smooth vector field and $x \in \mathcal{M} \subset \mathbb{R}^d$.

A linear action g is a symmetry of (1) if

$$v(gx) = g\, v(x) \tag{2}$$

for all $\mathcal{M}$. One says that v *commutes* with g or that v is *g-equivariant*. When v commutes with the set of group elements $g \in G$, the vector field v is said to be G-equivariant. The group G is said to be a *symmetry* of the dynamics if for every solution $x(\tau) = f^\tau(x)$, $g\,x(\tau)$ is also a solution. The finite-time flow $f^\tau(gx)$ through gx then satisfies the equivariance condition:

$$f^\tau(gx) = g f^\tau(x). \tag{3}$$

In physics literature the term *invariant* is most commonly used; for example, in Hamiltonian systems a symmetry is manifested as invariance of the Hamiltonian under the symmetry group action.

An element of a compact Lie group continuously connected to the identity can be written as

$$g(\theta) = e^{\theta\cdot\mathbf{T}}, \qquad \theta\cdot\mathbf{T} = \sum \theta_a \mathbf{T}_a, \quad a = 1, 2, \ldots, N, \tag{4}$$

where $\theta \cdot \mathbf{T}$ is a *Lie algebra* element, and θ_a are the parameters of the transformation. Repeated indices are summed throughout this section, and the dot product refers to a sum over Lie algebra generators. The Euclidean product of two vectors x, y is indicated by x-transpose times y, i.e., $x^T y = \sum_i^d x_i y_i$. Finite transformations $\exp(\theta \cdot \mathbf{T})$ are generated by sequences of infinitesimal steps of the form

$$g(\delta\theta) \simeq 1 + \delta\theta \cdot \mathbf{T}, \quad \delta\theta \in \mathbb{R}^N,\ |\delta\theta| \ll 1, \tag{5}$$

where $\mathbf{T}_a$, the *generators* of infinitesimal transformations, are a set of N linearly independent $[d \times d]$ anti-hermitian matrices, $(\mathbf{T}_a)^\dagger = -\mathbf{T}_a$, acting linearly on the d-dimensional state space $\mathcal{M}$. For $G \subset O(n)$ the generators can always be brought to the real, antisymmetric form $\mathbf{T}^T = -\mathbf{T}$. The flow induced by the action of the group on the state space point x is given by the set of N tangent fields

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j. \tag{6}$$

These tangent fields are always normal to the 'radial' vector x,

$$x^T t_a(x) = 0. \tag{7}$$

For an infinitesimal transformation (5) the G-equivariance condition (2) becomes

$$v(x) \simeq (1 - \theta\cdot\mathbf{T})\, v(x + \theta\cdot\mathbf{T}x) = v(x) - \theta\cdot\mathbf{T}\, v(x) + \frac{dv}{dx}\,\theta\cdot\mathbf{T}x.$$

Thus the infinitesimal, Lie algebra G-equivariance condition is

$$t_a(v) - A(x)\, t_a(x) = 0, \tag{8}$$

where $A = \partial v/\partial x$ is the stability matrix. The left-hand side,

$$\mathcal{L}_{t_a} v = \left(\mathbf{T}_a - \frac{\partial}{\partial y}(\mathbf{T}_a x)\right) v(y)\bigg|_{y=x}, \tag{9}$$

is known as the *Lie derivative* of the dynamical flow field v along the direction of the infinitesimal group-rotation-induced flow $t_a(x) = \mathbf{T}_a x$. The equivariance condition (8) states that the two flows, one induced by the dynamical vector field v and the other by the group tangent field t, commute if their Lie derivatives (or the Lie brackets or Poisson brackets) vanish.

Any representation of a compact Lie group G is fully reducible, and invariant tensors constructed by contractions of $\mathbf{T}_a$ are useful for identifying irreducible representations. The simplest such invariant is bilinear,

$$\mathbf{T}^T \cdot \mathbf{T} = \sum_\alpha C_2^{(\alpha)}\, \mathbb{1}^{(\alpha)}, \tag{10}$$

where $C_2^{(\alpha)}$ is the quadratic Casimir for the irreducible representation labeled α, and $\mathbb{1}^{(\alpha)}$ is the identity on the α-irreducible subspace, 0 elsewhere. The dot product of two tangent fields is thus a sum weighted by Casimirs,

$$t(x)^T \cdot t(x') = \sum_\alpha C_2^{(\alpha)} x_i\, \delta_{ij}^{(\alpha)} x'_j. \tag{11}$$

If x is not invariant (fixed under group actions), $t(x)^T \cdot t(x)$ is strictly positive. $t(x)^T \cdot t(x')$, however, can take either sign, or even vanish.

2.1. An example: complex Lorenz equations

Consider a complex generalization of Lorenz equations,

$$\dot{x} = -\sigma x + \sigma y, \qquad \dot{y} = (\rho - z)x - ay$$
$$\dot{z} = (xy^* + x^* y)/2 - bz, \tag{12}$$

where x, y are complex variables, z is real, while the parameters σ, b are real and $\rho = \rho_1 + i\rho_2$, $a = 1 - ie$ are complex. Recast in real variables, $x = x_1 + ix_2$, $y = y_1 + iy_2$, this is a set of five coupled ODEs:

$$\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma y_1, \qquad \dot{x}_2 = -\sigma x_2 + \sigma y_2 \\
\dot{y}_1 &= (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\
\dot{y}_2 &= \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\
\dot{z} &= -bz + x_1 y_1 + x_2 y_2.
\end{aligned} \tag{13}$$