Physica D 268 (2014) 50–58

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# On the stirring properties of the thermally-driven rotating annulus

R.J. Keane [a,b,*], P.L. Read [c], G.P. King [d,e]

[a] *Meteorologisches Institut der Universität München, Theresienstr. 37, 80333 München, Germany*
[b] *Deutscher Wetterdienst, Frankfurter Strasse 135, 63067 Offenbach, Germany*
[c] *Atmospheric, Oceanic & Planetary Physics, Clarendon Laboratory, Parks Road, Oxford OX1 3PU, UK*
[d] *Centro de Geofisica – IDL, Campo Grande, C8, University of Lisbon, 1749-016 Lisbon, Portugal*
[e] *Fluid Dynamics Research Centre, School of Engineering, University of Warwick, Coventry CV4 7AL, UK*

## HIGHLIGHTS

- The stirring properties of the thermally driven rotating annulus are investigated.
- The finite scale Lyapunov exponent (FSLE) is applied to the flow.
- The FSLE correctly identifies barriers between different flow regions.
- The Eulerian symmetry measure (ESM) is applied to the flow.
- The ESM more efficiently identifies some stirring properties of the flow.

## ARTICLE INFO

*Article history:*
Received 17 May 2013
Received in revised form
29 October 2013
Accepted 9 November 2013
Available online 16 November 2013
Communicated by H.A. Dijkstra

*Keywords:*
Lagrangian stirring
Eulerian symmetry measures
Geophysical fluid dynamics
Finite scale Lyapunov exponents
Thermally-driven rotating annulus

## ABSTRACT

The stirring properties of the thermally-driven rotating annulus have not been extensively studied, despite sustained interest in the stirring properties of various geophysical flows, and the wide applicability of the rotating annulus to geophysical problems. This paper takes important steps towards a thorough investigation of the stirring properties of thermally-driven rotating annulus flows, by demonstrating numerically the utility of two stirring measures for a parameter set yielding relatively simple flow conditions. The first measure is the finite scale Lyapunov exponent (FSLE), which has been successfully used to highlight the stirring properties of various geophysical flows. The second measure is the Eulerian symmetry measure, which has been far less widely used: this second measure does not provide such a detailed view of the stirring properties as the FSLE, but is far more efficient to calculate. Both measures are shown to have some success for the simple flow case studied, providing a strong foundation for further investigation into more complicated flows.

## 1. Introduction

The thermally-driven rotating annulus has long been used as an analogue for geophysical flows [1,2, for example]. It enables the basic dynamical properties of planetary flows to be studied in much greater detail than by observing the planetary flows themselves. It has recently been used to investigate synchronisation in climate dynamics [3,4], atmospheric predictability [5] and eddy parameterisation in oceanic and atmospheric flows [6].

Lagrangian stirring in fluid flows has been of geophysical interest over the last 20 years, in a diverse range of contexts [7–11]. It is therefore of great interest to study the Lagrangian stirring properties of the thermally-driven rotating annulus. This has been done by Sugata & Yoden [12], who numerically integrated passive tracer trajectories and showed how the heat transport from the outer to the inner boundary is effected predominately by transitions between clearly defined regions in the flow. Similar work was performed experimentally by Tajima et al. [13], which was in agreement with the numerical study. The purpose of this paper is to apply some new methods, developed subsequent to [12], to characterise the stirring properties of the rotating annulus flow and to investigate how these methods can identify the boundaries between the regions identified by [12].

This study follows on from Keane et al. [14], who investigated numerically the stirring properties of a simpler class of thermally-driven rotating annulus flows, made axisymmetric by artificially suppressing the development of three-dimensional instabilities. The previous study found the finite scale Lyapunov exponent

---

[*] Corresponding author at: Deutscher Wetterdienst, Frankfurter Strasse 135, 63067 Offenbach, Germany. Tel.: +49 6980622811.
*E-mail address:* richard.keane@dwd.de (R.J. Keane).

0167-2789/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physd.2013.11.004

[15, FSLE] to be particularly useful for quantifying stirring properties of the flow, and so it will be used in this study as a "standard" measure for quantifying the stirring properties of the annulus. The FSLE, as well as the closely related finite time Lyapunov exponent, has been used extensively in both geophysical and general fluid dynamics for identifying transport barriers [16,17], for illuminating the underlying structures of a flow, including identifying Lagrangian coherent structures [18–22], for assessing predictability with respect to soil moisture uncertainty [23], for quantifying the strength of the stirring in different regions of a flow [24] and for investigating the predictability of extreme events [25]. The FSLE is a practical version of the standard Lyapunov exponent, involving a finite-size initial separation and a finite-time trajectory integration.

A second measure is an Eulerian symmetry measure (ESM), introduced by Yannacopoulos et al. [26]. This measure does not require integration of Lagrangian tracer trajectories, relying instead on measuring the departure from certain symmetries of the velocity field that constrain the stirring. It was successfully applied by King et al. [27], who used it to measure the stirring properties of flows in the well-known Taylor–Couette system and their correlation with more established Lagrangian measures. However, it has so far not been applied to geophysical flows, and it makes sense to apply it to the rotating annulus flow to determine whether or not it can capture any aspects of the Lagrangian stirring properties without having to calculate Lagrangian trajectories.

An important part of this study is to compare the simpler ESM with the more reliable FSLE, and to determine what aspects of the stirring can be captured by the ESM and what aspects require the FSLE. Such comparison between Eulerian and Lagrangian measures of stirring has become of interest recently in geophysical fluid dynamics: for example, the Okubo Weiss measure has been compared with both the FSLE [17] and Lagrangian coherent structures [28,29], and Leung [30] has developed Eulerian methods for estimating the Lagrangian finite time Lyapunov exponent.

This paper continues with Section 2, which provides an overview of the rotating annulus flow that is numerically simulated here. The results are presented in Section 3, which shows how the FSLEs can identify different regions of the flow, and in Section 4, which shows how closely integrals of the ESMs correlate with integrals of the FSLEs. The results are summarised in Section 5, which also discusses future work that might build upon this study.

## 2. Case study

The flow considered is that of a viscous, thermally conducting fluid in a rotating annulus subject to horizontal differential heating. The two numerical codes used are a Navier–Stokes model described by Farnell & Plumb [31] and validated by Hignett et al. [1], and a modification of the particle path tracking code written by Rudman [32,27].

The codes are described by Keane et al. [14]: the main differences in this study are that the flows are no longer constrained to be axisymmetric, and that there is no time variation in the sidewall temperature forcing. The interpolation for the particle path tracking code is tricubic in space, and the number of grid points on which the velocity field is stored is 24 in $r$ and $z$ and 64 in $\vartheta$, on a staggered mesh with a tanh stretch in $r$ and $z$. The wave pattern 'drifts' with respect to the annulus in the azimuthal ($\vartheta$) direction: this drift is subtracted off the angular component of the velocity so that the tracer integrations are carried out in a reference frame in which the velocity is time independent.

The parameter values used in this study, and their meanings, are shown in Table 1.

**Table 1**
Parameter values used in this paper.

| Parameter | Meaning | Value |
|---|---|---|
| $a$ | Inner radius | 2.5 cm |
| $b$ | Outer radius | 8 cm |
| $d$ | Height | 14 cm |
| $\Omega$ | Rotation rate | 1.2 $\text{s}^{-1}$ |
| $T_b - T_a$ | Sidewall temperature difference | 4 K |
| $(T_b + T_a)/2$ | Mean sidewall temperature | 293 K |
| $\nu$ | Viscosity | $2.1163 \times 10^{-2}$ $\text{cm}^2\ \text{s}^{-1}$ |
| $\kappa$ | Thermal conductivity | $1.2623 \times 10^{-3}$ $\text{cm}^2\ \text{s}^{-1}$ |
| $\alpha$ | Thermal expansion coefficient | $-3.279 \times 10^{-4}$ $\text{K}^{-1}$ |
| $\rho_0$ | Background density | 1.048 g $\text{cm}^{-3}$ |

The flow is characterised by two dimensionless parameters – the 'thermal Rossby number' $\Theta$ and the Taylor number $\mathcal{T}$ [33,34] – defined as follows:

$$\Theta \equiv \frac{gd\alpha(T_b - T_a)}{\Omega^2(b-a)^2}, \qquad \mathcal{T} \equiv \frac{4\Omega^2(b-a)^5}{\nu^2 d} \tag{1}$$

where $g$ is the acceleration due to gravity. The values used in this study fall within the 'steady wave' regime, so that the flow is time independent and, as in [14], the number of phase space dimensions is three (here the three space dimensions).

### 2.1. Eulerian properties of the flow

The flow is governed by the Navier–Stokes equations, given as Eqs. (1)–(4) in [14], with time-derivatives equal to zero. The Eulerian fields have a wavenumber 4 disturbance in the azimuthal direction. The flow is shown in Figs. 1 and 2 and is characterised by a meandering azimuthal jet, which changes direction once in the vertical, surrounding helical vortices. The angular speed ($v$) field plots clearly display the meandering jet, as the peak in $v$ moves from one side to the other between the two vertical planes; it is also clear that the direction of the jet changes at about mid-height. In the temperature field, the nature of the isotherms reflects the balance between conduction and convection: conduction tends to make them vertical, while convection tends to cause them to become horizontal and stably stratified in the interior.

### 2.2. Lagrangian properties of the flow

The Lagrangian properties of the thermally-driven rotating annulus flow were investigated by Sugata & Yoden [12], who partitioned the flow into seven parts: upper- and lower-level meandering jets, anticyclonic and cyclonic trapped vortices and three boundary regions (lower-, inner- and outer-boundaries; their flow had no distinct upper-boundary region as they imposed a free upper surface). It is interesting to note that these regions were also identified here for slightly different flow parameters (as well as an eighth, upper-boundary region due to the rigid lid imposed in the present study). A brief investigation showed that the transitions between the regions also seem to be broadly similar, with a general anticlockwise vertical circulation (with trajectories passing from the outer-boundary, to the upper-boundary/upper-level-jet, to the inner-boundary, to the lower-boundary/lower-level-jet and then back to the outer boundary), and relatively long, but relatively few, residences in the trapped vortex regions. The transitions from and to the trapped vortices were mainly to and from the corresponding meandering jet. The trapped vortices were found to occur both inside and outside the meandering jet.

Instead of using Poincaré sections and detailed investigation of inter-regional transitions of trajectories, as in [12], the present study investigates the effectiveness of Finite Scale Lyapunov Exponents (FSLEs) in identifying the boundaries between the different regions. The results of this investigation are presented in the following section.