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Solitary waves in nematic liquid crystals

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HIGHLIGHTS

- Study solitary wave solutions of 2-D nonlocal NLS modeling nematicons.
- Show existence and symmetry properties of ground state solitary waves.
- Show existence of power threshold for negative energy solitary waves.
- Show decay of small power initial conditions.
- Compare infinite plane theory with numerical solution in finite domain.

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ABSTRACT

We study soliton solutions of a two-dimensional nonlocal NLS equation of Hartree-type with a Bessel potential kernel. The equation models laser propagation in nematic liquid crystals. Motivated by the experimental observation of spatially localized beams, see Conti et al. (2003), we show existence, stability, regularity, and radial symmetry of energy minimizing soliton solutions in $\mathbb{R}^2$. We also give theoretical lower bounds for the L^2-norm (power) of these solitons, and show that small L^2-norm initial conditions lead to decaying solutions. We also present numerical computations of radial soliton solutions. These solutions exhibit the properties expected by the infinite plane theory, although we also see some finite (computational) domain effects, especially solutions with arbitrarily small power.

1. Introduction

We study some basic properties of solitary waves in a nonlocal nonlinear Schrödinger (NLS) equation modeling the propagation of laser light in nematic liquid crystals. The model was proposed by Conti, Peccianti, and Assanto [1], who also conducted experiments and found stable optical solitons with a two-dimensional transverse profile. Other physical systems modeled by this or related nonlocal NLS equations are discussed in [2]. The stabilization of solitons and related lack of blow-up in the model is due to the nonlocality of the nonlinear interaction, and was predicted by earlier theoretical works, see [3,4]. More recent experiments examined this effect in other physical systems [5]. The regularizing effect of the nonlocal nonlinearity makes the liquid crystal system an interesting laboratory for studying two dimensional solitons, and there is considerable recent experimental and theoretical work on vortices [6,7], soliton interactions [8], multicolor solitons [9] and other related coherent structures [10].

In the present work we show the existence, regularity, and radial symmetry of energy minimizing solitons and compute radial solitons numerically. We also give analytically lower bounds for the L^2-norm (power) of energy minimizing solitons of negative energy. These thresholds involve best constants for the Gagliardo–Nirenberg and Hardy–Littlewood inequalities. It is possible that initial conditions with positive energy decay. While we do not settle this issue here, we use a different line of reasoning to show that initial conditions with sufficiently small L^2-norm decay.

The original model of [1] couples a Schrödinger equation for the evolution of the electric field amplitude to a nonlinear elliptic equation for the director field. The time variable is physically the distance along the optical axis. We here consider a common simplification of this model that leads to an NLS equation with a cubic Hartree-type nonlinearity on the plane, see [6,9]. The kernel of the Hartree nonlinearity is the two-dimensional Bessel potential (also known as a modified Bessel function). The particular NLS equation in two dimensions was discussed earlier in [11] where it was argued heuristically that it should have stable solitons, in contrast to the well known situation for the standard cubic NLS in two dimensions, where solitons are unstable and solutions can blow-up in finite time, see [12]. Turitsyn [4] uses a Gagliardo–Nirenberg inequality and energy conservation to argue

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that the H^1-norm of the solutions should stay bounded for all times. A simpler energy argument for bounded Hartree kernels appears in [13,14]. A rigorous version of the energy argument is also implicit in the work of Ginibre and Velo [3], see also [15], who consider cubic Hartree nonlinearities with more general kernels that include the one studied here.

Soliton solutions are obtained by minimizing the Hamiltonian of the nonlocal NLS over H^1 functions with fixed L^2-norm (power). The existence of minimizers is shown here by a concentration-compactness argument. We note that P.L. Lions [16] considered related quartic functionals with Hartree kernels. We further use elliptic regularity and rearrangement inequalities to see that the minimizer is a smooth radially symmetric decreasing function (up to translation and global phase change). The existence of constrained energy minimizers assumes that the L^2-norm is above a certain threshold, and we give two explicit lower bounds for this threshold. The idea is to bound below a ratio involving the quartic and quadratic parts of the energy, and the power. A similar ratio appears in the work of Weinstein [12] on the cubic NLS on the plane, although the present problem is closer to the situation in the discrete NLS [17]. Our power threshold estimate here involves best constants for the Gagliardo–Nirenberg inequality, and we also note an alternative bound involving constants for the Gagliardo–Nirenberg and Hardy–Littlewood inequalities.

We also include a fixed point argument in a space–time Lebesgue space that shows that initial conditions with small power decay. This proof uses the Strichartz estimates for the free Schrödinger evolution, and is similar to the one of [18] for the cubic power NLS on the plane. The decay argument gives a third bound for the minimum L^2-norm of H^1 solitons. A similar combination of absence of blow-up and decay for small solutions was also seen in the discrete cubic NLS in the two dimensional integer lattice, see [19].

We also present numerical computations of positive, decreasing radial solitons. The numerical study uses a finite circular domain with Dirichlet boundary conditions, and solitons are computed using the method of [20]. The numerical results are consistent with the existence of power threshold for negative energy solitons, but we also observe soliton-like solutions of arbitrarily small L^2-norm and positive energy. The existence of these small solutions can be explained by an abstract local bifurcation result, applied to the finite domain problem and its discretizations. We thus expect that only part of the calculated solution branch yields approximations to solitons of the $\mathbb{R}$ problem. The transition to the "spurious" part may involve collision with other solution branches, see [21] for such a scenario in the discrete NLS. This problem is left for further work.

The paper is organized as follows. In Section 2 we state our main theoretical results for the planar ($\mathbb{R}^2$) problem. In Section 3 we review the main results used in the proofs. In Section 4 we prove the existence, regularity, and symmetry properties of the minimizing solitons. In Section 5 we prove the power threshold and small amplitude decay theorems for the planar problem. In Section 6 we present the numerical results, and interpret them using the planar theory and a local bifurcation argument for the finite domain problem. In Section 7 we briefly discuss some further problems.

2. Soliton solutions of the nematicon equation

We consider the single-color nematicon equation in $\mathbb{R}^2$

$$iu_t + \frac{1}{2}D\Delta u + 2A\theta u = 0, \tag{2.1}$$

$$-\Delta\theta + m^2\theta = \frac{A}{\nu}|u|^2, \tag{2.2}$$

with constants $D, A, \nu > 0$, e.g. compare with [9]. The variable u represents the electric field envelope amplitude of an optical beam propagating through a nematic liquid crystal, while θ represents the angle of the director field of the liquid crystal.

The inhomogeneous elliptic Eq. (2.2) has a unique solution $\theta = G(|u|^2)$, with G a linear operator of convolution type, so that system (2.1), (2.2) is equivalent to equation

$$iu_t + \frac{1}{2}D\Delta u + 2AG(|u|^2)u = 0. \tag{2.3}$$

Taking the Fourier transform of (2.2) we have

$$\hat{\theta}_k = \frac{\hat{f}_k}{|k|^2 + m^2}, \quad \text{with } f = \frac{A}{\nu}|u|^2. \tag{2.4}$$

Thus G is convolution with the inverse Fourier transform of $A\nu^{-1}(|k|^2 + m^2)^{-1}$, and we have

$$\theta(x) = G(|u|^2)(x) = \frac{A}{\nu}\int_{\mathbf{R}^2} K_0(m|x-y|)|u(y)|^2 d^2y \tag{2.5}$$

where K_0 is the modified Bessel function, see [22], ch. III, or Bessel potential in $\mathbb{R}^2$ (up to constants), see [23], p.186. G is a bounded, self-adjoint operator in $L^2(\mathbb{R}^2, \mathbb{C})$.

The kernel $K_0 : \mathbb{R}^+ \to \mathbb{R}$ is positive, strictly decreasing and has the respective small and large r asymptotics

$$K_0(r) = \frac{1}{2\pi}\left(-\log r + (\log 2 - \gamma)\right) + O(r^2), \quad \text{as } r \to 0 \tag{2.6}$$

$$K_0(r) = \frac{1}{2\pi}\sqrt{\frac{\pi}{2r}}e^{-r}\left(1 + O(r^{-1})\right), \quad \text{as } r \to \infty, \tag{2.7}$$

with γ the Euler–Mascheroni constant, see [22], Ch. III–V. We also use the notation

$$K_{0,\mu}(r) = K_0(\mu r), \quad \mu, r > 0. \tag{2.8}$$

To avoid the singularity of K_0 at the origin, some authors have studied the nematicon system (2.3) using bounded kernels with a comparable fast decay at infinity, such as Gaussians, see [24,14], [13]. Most of the qualitative results below seem to apply to these models as well, see [2] for some differences.

The nematicon Eq. (2.3) with $\theta = G(|u|^2)$, G as above, can be also written formally as

$$u_t = -i\frac{\delta H}{\delta u^*}, \quad \text{with } H = \int_{\mathbb{R}^2}\left(\frac{D}{2}|\nabla u|^2 - A|u|^2 G(|u|^2)\right), \tag{2.9}$$

i.e. H is the *Hamiltonian* or *energy* of (2.3). Another conserved quantity of (2.3) is the *power* P, defined as

$$P(u) = \int_{\mathbb{R}^2}|u|^2, \tag{2.10}$$

see [25] for other conserved quantities.

In contrast to the two-dimensional cubic NLS whose solutions can blow up in finite time, see e.g. [26,12], the nonlocal analogue (2.3) with G as in (2.5) has solutions that exist for all times.

To state the simplest long time existence result we let $L^p(\mathbb{R}^N) = L^p(\mathbb{R}^N; \mathbb{C})$, $H^s(\mathbb{R}^2) = H^s(\mathbb{R}^2; \mathbb{C})$, $s \in \mathbb{R}$. We then have:

Theorem 2.1. *The initial value problem for (2.3) with $u(0) \in H^1(\mathbb{R}^2)$ has a unique solution $u \in C^0(\mathbb{R}; H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^2))$. Moreover, $\|u(t)\|_{H^1(\mathbb{R}^2)} \le M_0$ for some $M_0 > 0$, for all $t \in \mathbb{R}$.*

The local existence follows from a standard fixed point argument. Global existence follows by a conservation of energy argument using the idea of [4]. We give an abbreviated proof, since the argument is also implicit in [3], see also [15], ch. 6.