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Crack growth in polyconvex materials

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ABSTRACT

We discuss a model for crack propagation in an elastic body, where the crack path is described a priori. In particular, we develop in the framework of finite-strain elasticity a rate-independent model for crack evolution which is based on the Griffith fracture criterion. Due to the nonuniqueness of minimizing deformations, the energy-release rate is no longer continuous with respect to time and the position of the crack tip. Thus, the model is formulated in terms of the Clarke differential of the energy, generalizing the classical crack evolution models for elasticity with strictly convex energies. We prove the existence of solutions for our model and also the existence of special solutions, where only certain extremal points of the Clarke differential are allowed.

1. Introduction

In this work we discuss a model for crack propagation in an elastic body, where the crack path is prescribed a priori. Typical applications involve a body consisting of two parts that are glued together along a potential crack path. The evolution is assumed to be sufficiently slow such that inertial terms can be neglected, which is the quasistatic setting. Even more, we are interested in the rate-independent limit, which is relevant for cases, where the external loading via time-dependent forces is much slower than internal relaxation times. Thus, this paper also relates to the work in [1–3] where prescribed crack paths are considered for cohesive zone models describing delamination with partially debonded crack surfaces. However, in this work we follow [4–6] and restrict ourselves to brittle fracture, where only the not-yet-opened and the already-opened states are admitted for the crack such that the position of the crack tip determines all information about the crack. The evolution of the crack tip is assumed to follow the Griffith law, namely a crack does not move if the energy-release rate is less than the fracture toughness and it moves if the energy-release rate is larger, cf. e.g. [7–10] for work on Griffith criterion. We refer to [11] and the references therein for the physical background and numerical simulations. In particular our paper provides an existence result for a simplified version of the model in [11].

The novelty of the present work is that we allow for finite-strain elasticity in the bulk of the material. Thus, the elastic energy is nonconvex and for a given crack position there may be several minimizing deformations $\varphi : \Omega \to \mathbb{R}^2$ of the elastic energy. Moreover, the energy functional is no longer continuous on the set of admissible deformations as we impose the local invertibility constraint $\det \nabla\varphi > 0$ almost everywhere in Ω. We exploit the fact that the existence of energy-release rates for this case was established in [12]. However, in contrast to the work in [7–9,5,6,10] we are now faced with the difficulty that the energy-release rate is no longer continuous with respect to the time and the position of the crack tip, since it is defined via a minimization over the set of all possible minimizers for the current time and crack-tip position.

Following [13,1,6,14] we construct solutions for the rate-independent limit by a method of vanishing viscosity. However, our aim is to derive limit equations that describe the occurring limit solutions (also called approximable solutions) as precisely as

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possible. In this work we will obtain solutions called *local energetic solutions* which are the same as the *BV solution* defined in [14], except that here we are in a unidirectional setting ($\dot{s} \geq 0$) while there symmetric dissipation distances are used. Because of the jumps occurring it is useful to introduce *parameterized solutions* as used also in [13,14] (called parameterized metric solutions in the latter work). Since the present work allows for nonconvex elasticity the underlying (reduced) energy functional will only be Lipschitz continuous with points of non-differentiability that are locally nonconvex.

Thus, the above-mentioned local energetic model is formulated using the Clarke differential, which is the largest one among the different choices for the differentials at our disposal. We will also define corresponding *special local energetic* and *special parameterized solutions*, where only certain extremal points in the Clarke differential are allowed.

To be more specific, the set $\Omega \subset \mathbb{R}^2$ is the reference configuration of the elastic body, which is assumed to be a bounded Lipschitz domain. We denote by $t \in [0, T]$ the process time and by $s \in [s_0, s_1] \subset [0, L]$ the position of the crack tip. Here $\gamma : [0, T] \to \overline{\Omega}$ is the prescribed crack path in arc-length parameterization and we assume $\gamma \in C^{2,1}([0, L]; \mathbb{R}^2)$. For a given crack position s the set of admissible deformations is $W^{1,p}(\Omega_s; \mathbb{R}^2)$, where $\Omega_s = \Omega \setminus \{\gamma(\sigma) \mid \sigma \in [0, s]\}$. We define the *reduced energy functional* $\mathcal{I} : [0, T] \times [s_0, s_1] \to \mathbb{R}$ by minimizing the full energy functional with respect to the elastic deformation:

$$\mathcal{I}(t, s) = \min\left\{\int_{\Omega_s} W(\nabla\varphi)\,dx - \langle \ell(t), \varphi\rangle \mid \varphi \in W^{1,p}(\Omega_s; \mathbb{R}^2),\, (\varphi - g_{\mathrm{Dir}})\big|_{\Gamma_{\mathrm{Dir}}} = 0\right\}.$$

Under suitable technical assumptions we show that the mapping $(t, s) \mapsto \mathcal{I}(t, s) - \frac{\lambda}{2}(t^2 + s^2)$ is concave for a suitable $\lambda > 0$. Thus, for each point (t_*, s_*) all directional derivatives exist and determine the Clarke differential completely. In fact, in the λ-concave case there is a close relation between different notions of subdifferentials like the Clarke differential, the Fréchet differential and the subdifferential from convex analysis, cf. Section 3.1. In [12] it was shown that the total energy-release rate

$$\mathcal{G}(t, s) := -\partial_s^+\mathcal{I}(t, s) \geq 0$$

exists for all t and s, but we need additional one-sided continuity properties and semi-continuities of the one-sided partial derivatives $\partial_s^\pm\mathcal{I}$ and $\partial_t^\pm\mathcal{I}$. The concavity implies for the negative of the energy-release rates the estimates $\partial_s^+\mathcal{I}(t, s) \leq \partial_s^-\mathcal{I}(t, s)$, where inequality occurs due to different elastic minimizers. We define

$$\mathcal{G}^-(t, s) := -\partial_s^-\mathcal{I}(t, s)$$

satisfying

$$0 \leq \mathcal{G}^-(t, s) = \lim_{\delta\searrow 0}\mathcal{G}(t, s - \delta) \leq \mathcal{G}(t, s).$$

The fracture toughness is encoded in the continuous function $\kappa : [s_0, s_1] \to]0, \infty[$. Since our solutions will be non-decreasing the left-hand limit $s(t^-)$ and the right-hand limit $s(t^+)$ exist for all t and we define the continuity set $C(s) = \{t \in [0, T] \mid s(t^-) = s(t) = s(t^+)\}$. With this we obtain the jump set $J(s)$ and the differentiability set $D(s)$ as follows

$$J(s) = [0, T] \setminus C(s), \qquad D(s) = \{t \in [0, T] \mid \dot{s}(t) \text{ exists}\}.$$

A *local energetic solution* to the crack problem is a function $s \in BV([0, T]; [s_0, s_1])$ that satisfies for all $t \in [0, T]$ the following conditions

(a) s is non-decreasing;
(b) if $t \notin J(s)$, then $\kappa(s(t)) + \partial_s^-\mathcal{I}(t, s(t)) \geq 0$;
(c) if $\kappa(s(t)) + \partial_s^+\mathcal{I}(t, s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$;
(d) for all $t_* \in J(s)$ and all $s_* \in [s(t_*^-), s(t_*^+)]$ we have $\kappa(s_*) + \partial_s^+\mathcal{I}(t_*, s_*) \leq 0$.

Condition (a) is the unidirectionality (sometimes called irreversibility). Condition (b) is a kind of stability condition for rate-independent systems, namely $\mathcal{G}^-(t, s(t)) = -\partial_s^-\mathcal{I}(t, s(t)) \leq \kappa(s(t))$. This means that the smallest possible energy release cannot be bigger than the fracture toughness since otherwise the crack would have already moved further. Condition (c) is one part of the Griffith criterion, namely that the crack does not move if the release rate $\mathcal{G}(t, s(t))$ is less than the toughness $\kappa(s(t))$. Condition (d) states that along a jump the energy-release rate is at least as big as the toughness.

We will show in Section 4 that limits from vanishing-viscosity, time-incremental problems are in fact local energetic solutions. Actually the discrete solutions for the incremental problems are strictly related with the special local energetic solutions (cf. formula (4.4)), but in order to perform the limit passage as the time step goes to zero we have to also involve $\partial_s^-\mathcal{I}$ and therefore are able to derive local energetic solutions. However, as indicated via the example discussed in Section 4.3 there may still be too many solutions of this type. In fact, we conjecture that the limits constructed are always *special local energetic solutions*, which differ from the general local energetic solutions by replacing (b) by the stronger condition

(b_s) if $t \notin J(s)$, then $\kappa(s(t)) + \partial_s^+\mathcal{I}(t, s(t)) \geq 0$.

This leads to the exact Griffith criterion $\mathcal{G}(t, s(t)) = \kappa(s(t))$ along slowly moving cracks.

In Section 5 we finally show that special local energetic solutions exist. For this we use corresponding parameterized solutions. Moreover, for these solutions we establish the energy balance

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\,d\sigma + \mu(s, [t_1, t_2]) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t^-\mathcal{I}(\tau, s(\tau))\,d\tau$$

where, as in [6], $\mu(s, [t_1, t_2])$ denotes the extra energy losses along jumps at times $t \in [t_1, t_2]$, see (2.8).

Finally we emphasize that our local energetic solutions are quite different from the energetic solutions discussed in [7,8,15], as the energetic solutions always satisfy a global stability condition which is stronger than (b) and (c), but in return the jumps are considered as true jumps and nothing is said about the curve connecting the points $s(t^-)$ and $s(t^+)$ and (d) is not valid. However, the global stability enforces the energy balance (2.9) with $\mu \equiv 0$. See also the discussion in [6].

2. Set up of the model

In this section we collect all the assumptions on the data that will be satisfied throughout this paper.

The reference configuration is a bounded open subset of the plane, $\Omega \subset \mathbb{R}^2$, with Lipschitz boundary $\partial\Omega$. We assume that $\partial\Omega$ is the union of two disjoint subsets Γ_D and Γ_N, with $\mathscr{H}^1(\Gamma_D) > 0$, where $\mathscr{H}^1$ denotes the one-dimensional Hausdorff measure. On the Dirichlet part of the boundary Γ_D we impose a time-dependent boundary deformation $g_{\mathrm{Dir}}(t)$, while on the Neumann part Γ_N we prescribe surface forces $h(t)$.

The prescribed crack path is represented by a simple $C^{2,1}$-path (i.e., the second derivative is Lipschitz continuous) $\mathcal{C} \subset \overline{\Omega}$ with $\mathscr{H}^1(\mathcal{C}) =: L$ and let $\gamma : [0, L] \to \mathcal{C}$ be its arc-length parameterization. We assume that for every $s \in\ ]0, L[$ we have