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Hopf fibration: Geodesics and distances

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ABSTRACT

Here we study geodesics connecting two given points on odd-dimensional spheres respecting the Hopf fibration. This geodesic boundary value problem is completely solved in the case of three-dimensional sphere and some partial results are obtained in the general case. The Carnot–Carathéodory distance is calculated. We also present some motivations related to quantum mechanics.

1. Introduction

Sub-Riemannian geometry is proved to play an important role in many applications, e.g., in mathematical physics, geometric mechanics, robotics, tomography, neurosystems, and control theory. Sub-Riemannian geometry enjoys major differences from the Riemannian being a generalization of the latter at the same time, e.g., the notion of geodesic and length minimizer do not coincide even locally, the Hausdorff dimension is larger than the manifold topological dimension, the exponential map is never a local diffeomorphism. There exists a large amount of literature developing sub-Riemannian geometry. Typical general references are [1–4].

The interest in odd-dimensional spheres comes first of all from finite-dimensional quantum mechanics modeled over the Hilbert space $\mathbb{C}^n$ where the dimension n is the number of energy levels and the normalized state vectors form the sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. The problem of controlled quantum systems is basically the problem of controlled spin systems, which is reduced to the left- or right-invariant control problem on the Lie group $SU(n)$. In other words, these are problems of describing the sub-Riemannian structure of $\mathbb{S}^{2n-1}$ and the sub-Riemannian geodesics; see e.g., [5,3]. The special case $n = 2$ is well studied and the sub-Riemannian structure is related to the classical Hopf fibration; see, e.g., [6,7]. At the same time, the sub-Riemannian structure of $\mathbb{S}^3$ comes naturally from the non-commutative group structure of $SU(2)$ in the sense that two vector fields span the smoothly varying distribution of the tangent bundle, and their commutator generates the missing direction. The missing direction coincides with the Hopf vector field corresponding to the Hopf fibration. The sub-Riemannian geometry on $\mathbb{S}^3$ was studied in [8–11]; see also [12]. Explicit formulas for geodesics were obtained in [13] by

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solving the corresponding Hamiltonian system, in [10] from a variational equation, in [12] by exploiting the Lie theory and in [14] by applying the structure of the principle $\mathbb{S}^1$-bundle. Spectral analysis of the sub-Laplacian on $\mathbb{S}^3$ was provided in [8]. One of the important helping properties of odd-dimensional spheres is that there always exists at least one globally defined non-vanishing vector field.

Observe that $\mathbb{S}^3$ is compact and many properties and results of sub-Riemannian geometry differ from the standard nilpotent case, e.g., Heisenberg group or Engel group. In the case $\mathbb{S}^{2n-1}$, $n > 2$, we have no group structure and the main tool is the global action of the group U(1). For example, in our paper we explicitly show that any two points of $\mathbb{S}^3$ can be connected with an infinite number of geodesics.

Because of important applications, we start our paper with the description of n-level quantum systems and motivation given by Berry phases. Further we continue with general formulas for geodesics. Then we concentrate our attention on the *geodesic boundary value problem* finding all sub-Riemannian geodesics between two given points. In the case of $\mathbb{S}^{2n-1}$ we solve it for the points of the fiber and for $\mathbb{S}^3$ we solve it for arbitrary two points. The Carnot–Carathéodory distance is calculated.

2. *n*-level quantum systems

The mathematical formulation of quantum mechanics is based on concepts of pure and mixed states. A complex separable Hilbert space $\mathfrak{H}$ with Hermitian product $\langle\cdot,\cdot\rangle$ (or $\langle\cdot|\cdot\rangle$ in Dirac notations) is called the *state space*. The exact nature of this Hilbert space depends on the concrete system. For an n-level quantum system, $\mathfrak{H} = \mathbb{C}^n$ with the standard Hermitian product $\langle z|w\rangle = \sum_{j=1}^{n} z_j\bar{w}_j$. An *observable* is a self-adjoint linear operator acting on the state space. A *state* ρ is a special case of observable which is Hermitian $\rho^\dagger = \rho$, normalized by $\operatorname{tr}\rho = 1$, and positive $\langle w|\rho|w\rangle \geq 0$ for all vector $|w\rangle \in \mathfrak{H}$, where $\rho|w\rangle$ denotes the result of the action of the operator ρ on the vector $|w\rangle$. A *pure state* is the one-complex-dimensional projection operator $\rho = |z\rangle\langle z|$ onto the vector $|z\rangle \in \mathfrak{H}$, i.e., an operator satisfying $\rho^2 = \rho$. Other states are called *mixed states*.

The space of pure states is isomorphic to the projectivization $\mathfrak{H}_{\mathbb{P}}$ of the Hilbert space $\mathfrak{H}$. So equivalently we can define pure states as normalized vectors $\langle z|z\rangle = 1$ modulo a complex scalar $e^{i\theta}$, where θ is called a phase. In the case of the n-level quantum system $\mathfrak{H} = \mathbb{C}^n$, normalization and the phase factor allow us to represent the space of pure states as the complex projective space $\mathbb{CP}^{n-1} \equiv \mathbb{C}^n_{\mathbb{P}}$. The second operation of phase factorization is realized by the higher-dimensional Hopf fibration

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1},$$

where $\mathbb{S}^{2n-1}$ is the real $(2n-1)$-dimensional sphere embedded into $\mathbb{C}^n$, and the base space $\mathbb{CP}^{n-1}$ is the set of orbits of the action $\mathbb{S}^1$ on $\mathbb{S}^{2n-1}$.

In what follows, the pure states ρ are the elements of the projective space $\mathbb{CP}^{n-1}$, at the same time we use the notation of state $|z\rangle \in \mathbb{S}^{2n-1}$ to refer to a normalized representative of the phase-equivalence class ρ. The real dimension of the space of pure states is $2n-2$. The projective space $\mathbb{CP}^{n-1}$ endowed with the Kählerian Fubini–Study metric on $\mathbb{CP}^{n-1}$ becomes a metric space, in which this Riemannian metric is given by the real part of the Fubini–Study metric and it coincides with the push-forward of the standard Riemannian metric (the real part of the Hermitian one) on the $(2n-1)$-sphere by the Hopf projection.

In the case $n = 2$ the projective complex plane $\mathbb{CP}$ is isomorphic to the sphere $\mathbb{S}^2$ (called the Bloch sphere for two-level systems in physics), which is thought of as the set of orbits of the classical Hopf fibration

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2.$$

Each pair of antipodal points on the Bloch sphere corresponds to a mutually exclusive pair of states of the particle, namely, spin up and spin down. The Bloch sphere and the Hopf fibration describe the topological structure of a quantum mechanical two-level system; see [6,7]. The interest to two-level systems, an old subject, recently has gained a renewed interest due to recent progress in quantum information theory and quantum computation, where two-level quantum systems became qubits coupled in q-registers. A qubit state is represented up to its phase by a point on the Bloch sphere. The topology of a pair of entangled two-level systems is given by the Hopf fibration

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{HP},$$

where $\mathbb{H}$ is the space of quaternions and $\mathbb{HP} \cong \mathbb{S}^4$ is its projectivization; see [15]. Generally, for entangled n-level systems we have

$$\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n-1} \to \mathbb{HP}^{n-1}.$$

The underlying manifold for the Lie group SU(2) is $\mathbb{S}^3$. Considering the higher-dimensional group SU(n), we see that it acts on $\mathbb{S}^{2n-1}$. However, its dimension is $n^2 - 1 > 2n - 1$, $n > 2$, and its manifold only contains the invariant sphere $\mathbb{S}^{2n-1}$. Returning back to the information theory motivation, the relevant group for p-qubits is SU(2^p); see, e.g., [16,17].