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Generalized continuous frames constructed by using an iterated function system

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ABSTRACT

In this paper we give the definition of a generalized continuous frame for a separable Hilbert space. For such a generalization, after obtaining some basic results concerning these frames, we construct a class of generalized continuous frames by using an iterated function system.

1. Introduction

Let H be a separable Hilbert space. A countable family of elements $\{f_n\}_{n\in\mathbb{N}}$ in H is called a (discrete) frame if there are positive constants A, B such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

for all $f \in H$. A and B are called frame bounds. The sequence is called Bessel if the second inequality above holds. In this case, B is called the Bessel bound. Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series, and today they have applications in a wide range of areas. A frame can be considered as a generalized basis in the sense that every element in H can be written as a linear combination of the frame elements. A generalization of frames was proposed by Kaiser [2] and independently by Ali et al. [3]. These frames are known as continuous frames. Gabardo and Han in [4] called these frames "frames associated with measurable spaces". Askari-Hemmat et al. in [5] called them "generalized frames". For more details we refer the reader to [6,3,7,8,4,5,9,10].

In this paper we give the definition of a generalized continuous frame for a separable Hilbert space. If in the definition $\#I = 1$, the generalized continuous frame will be a continuous frame, and if, further, $\Omega := \mathbb{N}$ and μ is the counting measure, the continuous frame will be a discrete frame. The paper is organized as follows. In Section 2, we generalize the definition of a continuous frame to that of a generalized continuous frame and give some basic results concerning these frames. In Section 3, we introduce the iterated function system (IFS) and construct a class of generalized continuous frames using it. Throughout this paper, H denotes a complex separable Hilbert space.

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2. The definition and some basic results

Let B_H be the collection of all Bessel sequences in Hilbert space, H. Evidently, all finite sequences belong to B_H, and H is contained in B_H. Let I be an at most countable index set. Now we give the definition of a generalized continuous frame.

Definition 2.1. Let H be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to B_H,\ \omega \to \{f_i(\omega)\}_{i\in I}$ is called a generalized continuous frame with respect to (Ω, μ) if:

(1) F is weakly measurable, i.e., for all $f \in H, i \in I,\ \omega \to \langle f, f_i(\omega)\rangle$ is a measurable function on Ω;
(2) there exist positive constants A, B such that

$$A\|f\|^2 \le \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \le B\|f\|^2, \quad \forall f \in H. \tag{2.1}$$

The constant A and B are called generalized continuous frame bounds; F is called a tight generalized continuous frame if $A = B$. The mapping F is called Bessel if the second inequality in (2.1) holds. In this case, B is called the Bessel bound. If $\#I = 1$, F is a continuous frame, and if, further, μ is a counting measure and $\Omega := \mathbb{N}$, F is called a (discrete) frame. Moreover, it also follows from the first inequality in (2.1) that F is complete, i.e., $\overline{\mathrm{span}}\{\cup_{\omega\in\Omega}\{f_i(\omega)\}_{i\in I}\} = H$.

Using the same method as [9], we derive some basic results for a generalized continuous frame F with respect to (Ω, μ). First, the mapping $\Psi : H \times H \to \mathbb{C}$ defined by

$$\Psi(f, g) = \int_\Omega \sum_{i\in I} \langle f, f_i(\omega)\rangle \langle f_i(\omega), g\rangle \mathrm{d}\mu(\omega)$$

is well defined, is a sesquilinear form (i.e., linear in the first variable and conjugate-linear in the second variable) and is bounded. By the Cauchy–Schwarz inequality we get

$$\begin{aligned}
|\Psi(f, g)| &\le \int_\Omega \left| \sum_{i\in I} \langle f, f_i(\omega)\rangle \langle f_i(\omega), g\rangle \right| \mathrm{d}\mu(\omega) \\
&\le \int_\Omega \left(\sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i\in I} |\langle g, f_i(\omega)\rangle|^2 \right)^{\frac{1}{2}} \mathrm{d}\mu(\omega) \\
&\le \left(\int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}} \left(\int_\Omega \sum_{i\in I} |\langle g, f_i(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \right)^{\frac{1}{2}} \\
&\le B\|f\|\|g\|.
\end{aligned}$$

Hence, $\|\Psi\| \le B$. By Theorem 2.3.6 in [11] there exists a unique operator $S_F : H \to H$ such that $\Psi(f, g) = \langle S_F f, g\rangle$ for all $f, g \in H$ and, moreover, $\|\Psi\| = \|S_F\|$. Since $\langle S_F f, f\rangle = \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 \mathrm{d}\mu(\omega)$, we see that S_F is positive and $AI \le S_F \le BI$. Hence, S_F is invertible. We call S_F a generalized continuous frame operator of F and use the notation $S_F f = \int_\Omega \sum_{i\in I} \langle f, f_i(\omega)\rangle f_i(\omega) \mathrm{d}\mu(\omega)$. Thus, every $f \in H$ has the representations

$$f = S_F^{-1} S_F f = \int_\Omega \sum_{i\in I} \langle f, f_i(\omega)\rangle S_F^{-1} f_i(\omega) \mathrm{d}\mu(\omega),$$

$$f = S_F S_F^{-1} f = \int_\Omega \sum_{i\in I} \langle f, S_F^{-1} f_i(\omega)\rangle f_i(\omega) \mathrm{d}\mu(\omega).$$

Let λ be the counting measure and F be Bessel with the bound B. We define the following transform associated with F:

$$U_F : H \to L^2(\Omega \times I, \mu \times \lambda), \qquad U_F f(\omega, i) = \langle f, f_i(\omega)\rangle.$$

Its adjoint operators are given weakly by

$$U_F^* : L^2(\Omega \times I, \mu \times \lambda) \to H, \qquad U_F^* \varphi = \int_\Omega \sum_{i\in I} \varphi(\omega, i) f_i(\omega) \mathrm{d}\mu(\omega).$$

It follows that $S_F = U_F^* U_F$. The operator U_F^* is called a pre-frame operator or synthesis operator and U is called an analysis operator of F. Moreover, $\|U_F\| = \|U_F^*\| \le \sqrt{B}$.

Let $B(\omega)$ be a non-negative function defined on Ω such that

$$\sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 \le B(\omega)\|f\|^2, \quad \text{for almost all } \omega \in \Omega.$$

For example, take $B(\omega)$ to be the optimal Bessel bound of $F(\omega)$, i.e., the infimum over all Bessel bounds of $F(\omega)$. It is enough to check the Bessel condition in Definition 2.1 on a dense subset of H: