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Stochastic least-action principle for the incompressible Navier–Stokes equation

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We dedicate our paper with respect and affection to K.R. Sreenivasan, on the occasion of his 60th birthday.

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ABSTRACT

We formulate a stochastic least-action principle for solutions of the incompressible Navier–Stokes equation, which formally reduces to Hamilton's principle for the incompressible Euler solutions in the case of zero viscosity. We use this principle to give a new derivation of a stochastic Kelvin Theorem for the Navier–Stokes equation, recently established by Constantin and Iyer, which shows that this stochastic conservation law arises from particle-relabelling symmetry of the action. We discuss issues of irreversibility, energy dissipation, and the inviscid limit of Navier–Stokes solutions in the framework of the stochastic variational principle. In particular, we discuss the connection of the stochastic Kelvin Theorem with our previous "martingale hypothesis" for fluid circulations in turbulent solutions of the incompressible Euler equations.

1. Introduction

Alternative formulations of standard equations can be very illuminating and can cast new light on old problems. As just one example, consider how Feynman's path-integral solution of the Schrödinger equation enabled intuitive new approaches to difficult problems with many degrees-of-freedom, such as quantum electrodynamics and superfluid helium. In this same spirit, many different mathematical formulations have been developed for the equations of classical hydrodynamics, both ideal and non-ideal. Recently, Constantin and Iyer [1] have presented a very interesting representation of solutions of the incompressible Navier–Stokes equation by averaging over stochastic Lagrangian trajectories in the Weber formula [2] for incompressible Euler solutions. Their formulation is a nontrivial application of the method of stochastic characteristics, well known in pure mathematics [3] (Chapter 6), in theoretical physics [4,5] and in engineering modeling [6,7]. The characterization of the Navier–Stokes solutions in [1] is through a nonlinear fixed-point problem, since the velocity field that results from the average over stochastic trajectories must be the same as that which advects the fluid particles. Constantin and Iyer have shown that their stochastic representation implies remarkable properties of Navier–Stokes solutions in close analogy to those of ideal Euler solutions, such as a stochastic Kelvin Theorem for fluid circulations and a stochastic Cauchy formula for the vorticity field.

In this paper, we point out some further remarkable features of the stochastic Lagrangian formulation of [1]. Most importantly, we show that the nonlinear fixed-point problem that characterizes the Navier–Stokes solution is, in fact, a variational problem which generalizes the well-known Hamilton–Maupertuis least-action principle for incompressible Euler solutions [8]. We shall demonstrate this result by a formally exact calculation, at the level of rigor of theoretical physics. A more careful mathematical proof, with the set-up of relevant function spaces, precise definitions of variational derivatives, etc. shall be given elsewhere. Closely related stochastic variational formulations of incompressible Navier–Stokes solutions have been developed recently by others [9–11] and a detailed comparison with these approaches will also be made in future work.

Our variational formulation sheds some new light on a basic proposition of [1], the stochastic Kelvin Theorem which was established there for smooth Navier–Stokes solutions at any finite Reynolds number. We show that this result is a consequence of particle-relabelling symmetry of our stochastic action functional for Navier–Stokes solutions, in the same manner as the usual Kelvin Theorem arises from particle-relabelling symmetry of the standard action functional for Euler solutions [8]. This result strengthens the conjecture made by us in earlier work [12,13] that a "martingale property" of circulations should hold for generalized solutions of the incompressible Euler equations obtained in the zero-viscosity limit. Indeed, the stochastic variational principle for Navier–Stokes solutions considered in the present work is very closely similar to a stochastic least-action principle for generalized solutions of incompressible Euler equations that was developed by Brenier [14–16]. One of the arguments advanced for the "martingale property" in [12] was particle-relabelling symmetry

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in a Brenier-type variational formulation of generalized Euler solutions. That argument, however, did not distinguish an arrow of time, so that fluid circulations might satisfy the martingale property either forward or backward in time. It was subsequently argued in [13] that the backward-martingale property is the correct one, consistent with time-irreversibility in the limit of vanishing viscosity. The present work shows that a small but positive viscosity indeed selects the backward martingale property, as expected for a causal solution.

2. The action principle

The action principle formulated here for Navier–Stokes solutions involves *stochastic flows* [3]. The relevant flows are those which solve a *backward Ito equation*:

$$\begin{cases}\hat{d}_t\mathbf{x}^\varpi(\mathbf{a},t)=\mathbf{u}^\varpi(\mathbf{x}^\varpi(\mathbf{a},t),t)\mathrm{d}t+\sqrt{2\nu}\hat{d}\mathbf{W}^\varpi(t), \quad t<t_f\\ \mathbf{x}^\varpi(\mathbf{a},t_f)=\mathbf{a}.\end{cases} \tag{1}$$

Here $\mathbf{W}^\varpi(t),\ t\in[t_0,t_f]$ is a d-dimensional Brownian motion on a probability space $(\Omega,P,\mathcal{F})$ which is adapted to a two-parameter filtration $\mathcal{F}_t^{t'}$ of sub-σ-fields of $\mathcal{F}$, with $t_0\le t<t'\le t_f$. Thus, $\mathbf{W}^\varpi(s)-\mathbf{W}^\varpi(s')$ is $\mathcal{F}_t^{t'}$-measurable for all $t\le s<s'\le t'$. The constant ν that appears in the amplitude of the white-noise term in the SDE (1) will turn out to be the kinematic viscosity in the Navier–Stokes equation. Note that, for such an additive noise as appears in (1), the (backward) Ito and Stratonovich equations are equivalent.

In order to describe the space of flow maps which appear in the action principle, we must make a few slightly technical, preliminary remarks. The random velocity field $\mathbf{u}^\varpi(\mathbf{r},t)$ in Eq. (1) is assumed to be smooth and, in particular, continuous in time, as well as adapted to the filtration $\mathcal{F}_t^{t_f},\ t<t_f$ backward in time. It then follows from standard theorems (e.g. see Corollary 4.6.6 of [3]) that the solution $\mathbf{x}^\varpi(\mathbf{a},t)$ of (1) is a backward semi-martingale of flows of diffeomorphisms. Conversely, any backward semi-martingale of flows of diffeomorphisms has a backward Stratonovich random infinitesimal generator $\mathring{\mathbf{F}}^\varpi(\mathbf{r},t)$ which is a spatially-smooth backward semi-martingale (e.g. see Theorem 4.4.4 of [3]). The class of such flows for which the martingale part of the generator is $\sqrt{2\nu}\mathbf{W}^\varpi(t)$ and for which the bounded-variation part of the generator is absolutely-continuous with respect to dt coincides with the class of solutions of equations of form (1), for all possible choices of $\mathbf{u}^\varpi(\mathbf{r},t)$. Clearly, the random fields $\mathbf{u}^\varpi(\mathbf{r},t)$ and $\mathbf{x}^\varpi(\mathbf{a},t)$ uniquely determine each other. We consider here the incompressible case, where $\mathbf{u}^\varpi(\mathbf{r},t)$ is divergence-free and $\mathbf{x}^\varpi(\mathbf{a},t)$ is volume-preserving a.s.

The action is defined as a functional of the backward-adapted random velocity fields $\mathbf{u}^\varpi(\mathbf{r},t)$—or, equivalently, of the random flow maps $\mathbf{x}^\varpi(\mathbf{a},t)$—by the formula

$$S[\mathbf{x}]=\int P(\mathrm{d}\varpi)\int_{t_0}^{t_f}\mathrm{d}t\int \mathrm{d}^d r\,\frac{1}{2}|\mathbf{u}^\varpi(\mathbf{r},t)|^2 \tag{2}$$

when this is well defined and as $+\infty$ otherwise. The *variational problem* (VP) is to find a stationary point of this action such that $\mathbf{x}^\varpi(\mathbf{a},t_f)=\mathbf{a}$ and $\mathbf{x}^\varpi(\mathbf{a},t_0)=\boldsymbol{\varphi}^\varpi(\mathbf{a})$ for $P-a.e.\ \varpi$, where $\boldsymbol{\varphi}^\varpi(\mathbf{a})$ is a given random field of volume-preserving diffeomorphisms of the flow domain. It is interesting that this problem is very similar to that considered by Brenier [14–16] for generalized Euler solutions. The above problem leads instead to the incompressible Navier–Stokes equation, in the following precise sense:

Proposition 1. *A stochastic flow $\mathbf{x}^\varpi(\mathbf{a},t)$ which satisfies both the initial and final conditions is a solution of the above variational problem if and only if $\mathbf{u}^\varpi(\mathbf{r},t)$ solves the incompressible Navier–Stokes equation with viscosity $\nu>0$*

$$\partial_t\mathbf{u}^\varpi+(\mathbf{u}^\varpi\cdot\nabla)\mathbf{u}^\varpi=-\nabla p^\varpi+\nu\Delta\mathbf{u}^\varpi, \quad P\text{-a.s.} \tag{3}$$

where kinematic pressure p^ϖ is chosen so that $\nabla\cdot\mathbf{u}^\varpi=0$.

Proof. Making a variation $\delta\mathbf{u}^\varpi(\mathbf{r},t)$ in the random velocity field, the Eq. (1) becomes

$$\begin{cases}\hat{d}_t\delta\mathbf{x}^\varpi(\mathbf{a},t)\\ \quad=[\delta\mathbf{x}^\varpi(\mathbf{a},t)\cdot\nabla_r\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)+\delta\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)]\mathrm{d}t, \quad t<t_f\\ \delta\mathbf{x}^\varpi(\mathbf{a},t_f)=\mathbf{0}.\end{cases} \tag{4}$$

Since the VP requires that $\mathbf{x}^\varpi(\mathbf{a},t_0)=\boldsymbol{\varphi}^\varpi(\mathbf{a})$, one can only consider variations such that, also, $\delta\mathbf{x}^\varpi(\mathbf{a},t_0)=\mathbf{0}$. (We shall consider below an alternative approach with a Lagrange multiplier that permits unconstrained variations.) This equation may also be written as

$$\hat{d}_t\delta\mathbf{x}^\varpi(\mathbf{a},t)-(\nabla_r\mathbf{u}^\varpi(\mathbf{x}^\varpi,t))^\top\delta\mathbf{x}^\varpi(\mathbf{a},t)\mathrm{d}t \\ =\delta\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t \tag{5}$$

for $t<t_f$ and then easily solved by Duhamel's formula (backward in time) to give $\delta\mathbf{x}^\varpi(\mathbf{a},t)$ in terms of $\delta\mathbf{u}^\varpi(\mathbf{r},t)$. Since the martingale term vanished under variation, the process $\delta\mathbf{x}^\varpi(\mathbf{a},t)$ is of bounded variation and clearly adapted to the backward filtration $\mathcal{F}_t^{t_f},\ t<t_f$. Conversely, any such flow variation will determine the corresponding velocity variation $\delta\mathbf{u}^\varpi(\mathbf{r},t)$ by Eq. (5) directly. Lastly, note that the volume-preserving condition $\det(\nabla_a\mathbf{x}^\varpi(\mathbf{a},t))=1$ becomes

$$\nabla_r\cdot\delta\mathbf{x}^\varpi(\mathbf{a}^\varpi,t)=0 \tag{6}$$

under variation, where $\mathbf{a}^\varpi(\mathbf{r},t)$ is the "back-to-labels" map inverse to the flow map $\mathbf{x}^\varpi(\mathbf{a},t)$. Because these maps are diffeomorphisms, we see that the Eulerian variation of the flow map, $\delta\bar{\mathbf{x}}^\varpi(\mathbf{r},t)\equiv\delta\mathbf{x}^\varpi(\mathbf{a}^\varpi(\mathbf{r},t),t)$, is an arbitrary divergence-free field.

With these preparations, we obtain for the variation of the action (2):

$$\begin{aligned}\delta S[\mathbf{x}]&=\int P(\mathrm{d}\varpi)\int_{t_0}^{t_f}\mathrm{d}t\int\mathrm{d}^d r\,\mathbf{u}^\varpi(\mathbf{r},t)\cdot\delta\mathbf{u}^\varpi(\mathbf{r},t)\\ &=\int P(\mathrm{d}\varpi)\int d^d a\int_{t_0}^{t_f}\mathbf{u}^\varpi(\mathbf{x}^\varpi(\mathbf{a},t),t)\\ &\quad\cdot\left[\hat{d}_t\delta\mathbf{x}^\varpi(\mathbf{a},t)-\delta\mathbf{x}^\varpi(\mathbf{a},t)\cdot\nabla_r\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t\right]\\ &=-\int P(\mathrm{d}\varpi)\int d^d a\int_{t_0}^{t_f}\left[\hat{d}_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\right.\\ &\quad\left.+\nabla_r\left(\frac{1}{2}|\mathbf{u}^\varpi|^2\right)\Big|_{\mathbf{x}^\varpi}\mathrm{d}t\right]\cdot\delta\mathbf{x}^\varpi(\mathbf{a},t).\end{aligned} \tag{7}$$

In the second line we employed (5). In the third line we integrated by parts, using the facts that $\delta\mathbf{x}^\varpi(\mathbf{a},t_f)=\delta\mathbf{x}^\varpi(\mathbf{a},t_0)=0$ and that $\delta\mathbf{x}^\varpi(\mathbf{a},t)$ is a bounded-variation process, so that the quadratic variation vanishes: $\hat{d}_t\langle\mathbf{u}^\varpi(\mathbf{x}^\varpi,t),\delta\mathbf{x}^\varpi(\mathbf{a},t)\rangle=0$. We note that the final gradient term vanishes, because $\delta\bar{\mathbf{x}}^\varpi(\mathbf{r},t)$ is divergence-free. We can evaluate the remaining term using the chain rule

$$\hat{d}_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t) \\ =\partial_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t+(\mathbf{x}^\varpi(\mathbf{a},\circ\hat{\mathrm{d}}t)\cdot\nabla)\mathbf{u}^\varpi(\mathbf{x}^\varpi,t), \tag{8}$$

in terms of the backward Stratonovich differential. This result can also be written using Ito calculus. Calculating from (1) and (8) the quadratic variation

$$\sqrt{2\nu}\hat{d}_t\langle W_j^\varpi(t),\partial_{x_j}\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\rangle=2\nu\Delta\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)\mathrm{d}t,$$

one obtains the backward Ito equation

$$\hat{d}_t\mathbf{u}^\varpi(\mathbf{x}^\varpi,t)=[\partial_t\mathbf{u}^\varpi+(\mathbf{u}^\varpi\cdot\nabla)\mathbf{u}^\varpi-\nu\Delta\mathbf{u}^\varpi](\mathbf{x}^\varpi,t)\mathrm{d}t \\ +\sqrt{2\nu}(\hat{d}\mathbf{W}^\varpi(t)\cdot\nabla)\mathbf{u}^\varpi(\mathbf{x}^\varpi,t). \tag{9}$$