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Invariant fourth root Finsler metrics on the Grassmannian manifolds

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ABSTRACT

Fourth root metrics are a special and important class of Finsler metrics, which have been applied to physics. In this paper, we study invariant fourth root Finsler metrics on the Grassmannian manifolds $SO(p+q)/SO(p) \times SO(q)$. By using the results from the theory of invariant polynomials of Lie groups, we obtain a complete classification of such metrics. Further, some invariant $2m$-th root Finsler metrics are also given.

1. Introduction

The m-th root metric defined by the fundamental function

$$F = \sqrt[m]{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \cdots y^{i_m}} \tag{1.1}$$

was first studied by Shimada [1] (see also [2–6]). It is a natural generalization of the Riemannian metric. In fact, in his famous lecture in 1854, Riemann mentioned the fourth root metrics as the next simplest case other than the second root metrics (the latter are now called Riemannian metrics). Recently, fourth root metrics have been taken as a model of space-time in physics (see, e.g. [6]). Li and Shen studied the projectively flat fourth root Finsler metrics (see [7]) and obtained some interesting results.

The purpose of this paper is to initiate the study of invariant fourth root metrics on homogeneous manifolds. Let G/H be a reductive homogeneous space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, i.e., $Ad(h)\mathfrak{m} \subset \mathfrak{m}, \forall h \in H$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively. Then the G-invariant Finsler metric is in one-to-one correspondence with the Minkowski norm on $\mathfrak{m}$ satisfying (see [8])

$$F(Ad(h)y) = F(y), \quad \forall h \in H,\ y \in \mathfrak{m}. \tag{1.2}$$

If F is an invariant m-th root Finsler metric on G/H, then $f = F^m$ is an $Ad(H)$-invariant positive polynomial function of degree m on $\mathfrak{m}$. Therefore m must be even. Hence any invariant m-th root Finsler metric must be reversible. For $m = 2$,

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the invariant quadratic root Finsler metrics are invariant Riemannian metrics. It is well known that there exist invariant Riemannian metrics on homogeneous spaces G/H when H is compact. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be an invariant Riemannian metric. Then $F = \sqrt[2n]{\alpha^{2n}}$ is an invariant $2n$-th root metric and is Riemannian. So we are more concerned with the non-Riemannian metrics of the form (1.1). By (1.2), in order to construct an invariant m-th root Finsler metric on G/H, we need to accomplish the following two steps:

1. To find all the H-invariant polynomials of degree m on $\mathfrak{m}$;
2. To determine which one can define a Minkowski norm on $\mathfrak{m}$.

In this paper, we investigate invariant fourth root Finsler metrics on the real Grassmannian manifolds $SO(p+q)/SO(p) \times SO(q)$. Using the first fundamental theorem of the invariant theory of Lie groups for $SO(p)$, we prove the following

Theorem 1.1. *For $G/H = SO(p+q)/SO(p) \times SO(q)$ with $p \geq q$ and $p+q \geq 3$, we have*

(1) *If $q = 1$, then there does not exist any invariant non-Riemannian fourth root Finsler metrics on G/H.*
(2) *If $p = q = 2$, then the isotropic representation is reducible and there exist infinitely many different invariant non-Riemannian fourth root Finsler metrics on G/H.*
(3) *If $p > 2, q \geq 2$, then the isotropic representation is irreducible and there exist infinitely many different invariant non-Riemannian fourth root Finsler metrics on G/H.*

For $q = 1$, the coset space is just $S^n = SO(n+1)/SO(n)$. It is well-known that the action of the isotropy representation of this coset space is transitive on the unit sphere of the tangent space at the origin with respect to the standard Riemannian metric. Hence any G-invariant Finsler metric on G/H must be a positive multiple of the standard Riemannian metric. This proves (1). While (2), resp. (3), follows from Theorem 4.2, resp. Theorem 5.5.

We will also construct all the invariant fourth root Finsler metrics in the above cases from (2) and (3), using different methods. In the first case, we obtain the classification by means of the decomposition of the isotropic representation and the invariant Riemannian metrics, see Section 4. This is a general result, namely, in a reductive homogeneous manifold G/H with H compact, if the isotropy representation of H in the tangent space $T_o(G/H)$, where o is the origin, is not irreducible, then there exist G-invariant non-Riemannian fourth root Finsler metrics on G/H. In the last case, the isotropy representation is irreducible, and we obtain the classification through developing an invariant theory. It should be noted that the invariant fourth root Finsler metrics on two-dimensional homogeneous spaces have been studied by Gorbatsevich (see [9]).

2. Preliminaries

In this section, we recall some notions in Finsler geometry.

Definition 2.1. Let M be a finite dimensional smooth manifold. A function $F : TM \to [0, +\infty)$ is a Finsler structure if it satisfies

(F1) F is C^∞ on the slit tangent bundle $TM \setminus 0$,
(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $x \in M$, $y \in T_xM$ and $\lambda > 0$,
(F3) For every $y \in T_xM \setminus 0$, the quadratic form

$$g_{x,y}(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{t=s=0}, \quad \forall u, v \in T_xM$$

is positive definite.

In this case, (M, F) is called a Finsler manifold. F is called reversible if $F(x, -y) = F(x, y)$ holds for all $x \in M$ and $y \in T_xM$.

Definition 2.2. A Finsler metric F is called a $2m$-th root metric if F is defined by the following form

$$F = \sqrt[2m]{a_{i_1i_2\cdots i_{2m}}(x)y^{i_1}y^{i_2}\cdots y^{i_{2m}}}.$$

Let T denote the $2m$-th power of a $2m$-th root metric F. Then

$$F = T^{\frac{1}{2m}},$$

$$F_{y^i} = \frac{1}{2m}T^{\frac{1}{2m}-1}T_{y^i},$$

$$F_{y^iy^j} = \frac{1}{4m^2}T^{\frac{1}{2m}-2}((1-2m)T_{y^i}T_{y^j} + 2mTT_{y^iy^j}),$$

$$g_{ij} = F_{y^i}F_{y^j} + FF_{y^iy^j} = \frac{1}{2m^2}T^{\frac{1}{m}-2}((1-m)T_{y^i}T_{y^j} + mTT_{y^iy^j}). \tag{2.3}$$

By the positivity of F, we get

Proposition 2.3. *g_{ij} is positive definite if and only if $(1-m)T_{y^i}T_{y^j} + mTT_{y^iy^j}$ is positive definite.*