Journal of Geometry and Physics 61 (2011) 172–179

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

The Riemann–Roch theorem and zero-energy solutions of the Dirac equation on the Riemann sphere

Geoffrey Lee *

20 Arlmont Drive, Kensington, CA 94707, USA

ARTICLE INFO

Article history:
Received 17 September 2009
Received in revised form 6 September 2010
Accepted 30 September 2010
Available online 8 October 2010

MSC:
14H81
14C40

Keywords:
Riemann–Roch theorem
The two-dimensional Dirac equation in a magnetic field
Zero modes
Riemann surfaces

ABSTRACT

In this paper, we revisit the connection between the Riemann–Roch theorem and the zero-energy solutions of the two-dimensional Dirac equation in the presence of a delta-function-like magnetic field. Our main result is the resolution of a paradox—the fact that the Riemann–Roch theorem correctly predicts the number of zero-energy solutions of the Dirac equation despite counting what seem to be functions of the wrong type.

1. Introduction

The cross-fertilization between mathematics and physics has proven fruitful over the history of science. Often important results in one field generate new ideas in others. For example, the seminal Atiyah–Singer index theorem in mathematics has spawned many novel directions of research in physics due to its connection with the Dirac equation with a background magnetic field. Examples include: string theory, particle theory, condensed matter theory, etc. While solving the Dirac equation with magnetic fields over arbitrary Riemann surfaces is an arduous task, the Riemann–Roch theorem, which counts functions of a particular type (namely, meromorphic functions), correctly predicts the analytic index of the Dirac Hamiltonian. When the magnetic field is sufficiently strong, this index agrees with the multiplicity of the zero-energy solution. Therefore it is natural to suspect that the Riemann–Roch theorem can yield information for the $E = 0$ eigenfunctions of the solutions to the Dirac equation.

The celebrated Riemann–Roch theorem [1] deals with functions that are analytic everywhere except at a finite set of poles on a closed and compact Riemann surface X (a Riemann surface is a smooth, orientable surface). These functions of interest are known as meromorphic functions. The characteristics of meromorphic functions are summarized by a "divisor" $D = \sum_{p \in X} n_p\, p$, where $\{p\}$ is a finite (and therefore discrete) set of points on X and n_p are integers [1]. The vector space $L(D)$, consisting of meromorphic functions that have poles of order $\le |n_p|$ at the points with $n_p < 0$ and zeros of order $\ge n_p$ at the points with $n_p \ge 0$, is the entity of interest. The Riemann–Roch theorem states that $\dim(L(D)) - \dim(L(K - D)) = \deg(D) + 1 - g$ with K any "canonical divisor", g is the genus of X and $\deg(D) = \sum_{p \in X} n_p$, the degree of the divisor. The concept of the canonical divisor is irrelevant for the purpose of this paper; this is because for divisors with sufficiently large

* Corresponding address: 1804 Harvard Yard Mail Center, Cambridge, MA 02138, USA.
E-mail address: geoffrey.y.lee@gmail.com.

0393-0440/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2010.09.023

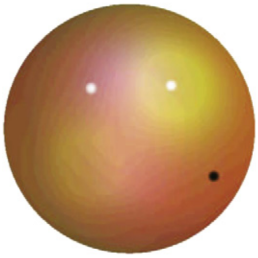

Fig. 1. (Color online) A graphical representation of a divisor. The white and black points denote poles and zeros respectively.

degree, $\dim(L(K-D)) = 0$, and hence K does not enter the consideration. This is the situation that we will be focusing on; however for further reading about canonical divisors, we refer the interested reader to Miranda [1].

In physics it is known that the Riemann–Roch theorem is connected to the analytical index [2] of the two-dimensional Dirac equation under a constant magnetic field [3,4]. When the total number of flux quanta[1] is sufficiently big, the latter reduces to the degeneracy of the $E = 0$ energy level. In this paper we show that this connection is quite subtle. First of all, meromorphic functions can diverge while the solutions[2] of the Dirac equation in uniform magnetic field do not. This discrepancy can easily be removed by deforming the magnetic field so that

$$B(\vec{r}) = \sum_i 2\pi n_i \delta(\vec{r} - \vec{r}_i) \tag{1}$$

while maintaining the total magnetic flux. After such deformation it is possible for the $E = 0$ solution of the Dirac equation to diverge at a subset of $\{\vec{r}_i\}$.[3] The fact that such deformation preserves the analytic index of the Dirac operator is a consequence of the Atiyah–Singer index theorem [2]. However a discrepancy remains even after this deformation, namely, the solutions of the Dirac equation depend on both z and its conjugate $\bar{z}$; these functions are not meromorphic. In this paper we examine and resolve this discrepancy when X is the Riemann sphere. This case is particularly simple because for any divisor D with $\deg(D) \geq 0$ on the Riemann sphere, $\dim(L(K-D)) = 0$. Moreover, this is a relevant example for realistic physical systems. This is because $\deg(D)$ represents the number of magnetic flux quanta which is usually much greater than 1. Nevertheless, we believe that the idea exposed here can be generalized to other Riemann surfaces with higher genera.

2. The Riemann–Roch problem on the sphere

For the sake of completeness, in this section the mathematics of the Riemann–Roch problem will be introduced.

2.1. Divisors

In order to assist in the study of meromorphic functions, mathematicians introduce the concept of a divisor. (The following definitions can be found in, e.g., Ref. [1].)

Definition 2.1. A divisor (Fig. 1) is a formal finite linear combination of points of X with integral coefficients, e.g. $D = \sum_i n_i p_i$. The collection of divisors on a surface X, denoted as $\mathrm{Div}(X)$, form an abelian group with the additive group operation defined as though distinct points were linearly independent vectors. Mathematicians define a partial ordering on $\mathrm{Div}(X)$ by defining $D \geq 0$ if all its coefficients are nonnegative. The degree of the divisor $D = \sum_i n_i p_i$ is defined as $\sum_i n_i$.

Divisors are of importance in the study of Riemann surfaces because the set of zeros and poles of a meromorphic function on a surface X can be associated with a divisor.

Definition 2.2. Let f be a nonzero meromorphic function on a compact Riemann surface X. Then the divisor of f is defined to be $\mathrm{div}(f) = \sum_{p \in X} n_p p$ where n_p is the order of the zero or pole of f at p. By convention, n_p is positive when f has a zero at p and negative when f has a pole at p. Such divisors are called principal divisors.

It is important to note that such a divisor is always well defined because a meromorphic function on a compact Riemann surface can only have finitely many zeros and poles [5].

---

1 The number of flux quanta is defined as Φ/Φ_0 where Φ is the total flux passing through the Riemann surface and Φ_0 is the Dirac flux quantum. We shall choose the natural unit and fix the charge of the particle so that $\Phi_0 = 2\pi$.

2 In the rest of this paper the phrase "solutions of the Dirac equation" is used to denote the $E = 0$ solutions of the time-independent Dirac equation in the presence of magnetic field.

3 The reader might wonder whether deforming the magnetic flux to make divergent solutions spoils square integrability. The fact is that the Dirac equation always is a low energy effective theory, and hence has a short distance cutoff. This cutoff rounds off the divergence.