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Comparing two approaches to the K-theory classification of D-branes

Fabio Ferrari Ruffino *, Raffaele Savelli

International School for Advanced Studies (SISSA/ISAS), Via Beirut 2, I-34014, Trieste, Italy
Istituto Nazionale di Fisica Nucleare (INFN), sezione di Trieste, Italy

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ABSTRACT

We consider the two main classification methods of D-brane charges via K-theory, in type II superstring theory with vanishing B-field: the Gysin map approach and the one based on the Atiyah–Hirzebruch spectral sequence. Then, we find out an explicit link between these two approaches: the Gysin map provides a representative element of the equivalence class obtained via the spectral sequence. We also briefly discuss the case of rational coefficients, characterized by a complete equivalence between the two classification methods.

1. Introduction

K-theory provides a good tool to classify D-brane charges in type II superstring theory [1,2]. In the case of vanishing B-field, there are two main approaches in the literature. The first one consists of applying the Gysin map to the gauge bundle of the D-brane, obtaining a K-theory class in space–time [3]. This approach is motivated by the Sen conjecture, stating that a generic configuration of branes and antibranes with gauge bundle is equivalent, via tachyon condensation, to a stack of coincident space-filling brane–antibrane pairs equipped with an appropriate K-theory class [4]. The second approach consists of applying the Atiyah–Hirzebruch spectral sequence (AHSS, [5]) to the Poincaré dual of the homology class of the D-brane: such a sequence rules out some cycles affected by global world-sheet anomalies, e.g. Freed–Witten anomaly [6], and quotients out some cycles which are actually unstable, e.g. MMS-instantons [7]. We assume for simplicity that the space–time and the D-brane world volumes are compact. For a given filtration of the space–time $S = S^{10} \supset S^9 \supset \cdots \supset S^0$, the second step of AHSS is the cohomology of S, i.e. $E_2^{p,0}(S) \simeq H^p(S, \mathbb{Z})$, while the last step of AHSS is given by (up to canonical isomorphism)

$$E_\infty^{p,0}(S) \simeq \frac{\mathrm{Ker}\big(K^p(S) \longrightarrow K^p(S^{p-1})\big)}{\mathrm{Ker}\big(K^p(S) \longrightarrow K^p(S^p)\big)}.$$

Hence, given a D-brane world-volume WY_p of codimension $10 - (p+1) = 9 - p$, with gauge bundle $E \to WY_p$ of rank q, if the Poincaré dual of WY_p in S survives until the last step of AHSS, it determines a class $\{\mathrm{PD}_S[q \cdot WY_p]\} \in E_\infty^{9-p,0}(S)$ whose representatives belong to $\mathrm{Ker}(K^{9-p}(S) \longrightarrow K^{9-p}(S^{8-p}))$.

These two approaches give different information, in particular AHSS does not take into account the gauge bundle: the aim of the present work is to relate them. We briefly anticipate the result. For a Dp-brane with world-volume $WY_p \subset S$ and gauge bundle $E \to WY_p$ of rank q, let $i : WY_p \to S$ be the embedding and $i_! : K(WY_p) \to K^{9-p}(S)$ the Gysin map. We will

* Corresponding author at: International School for Advanced Studies (SISSA/ISAS), Via Beirut 2, I-34014, Trieste, Italy.
E-mail addresses: ferrariruffino@gmail.com (F. Ferrari Ruffino), savelli@sissa.it (R. Savelli).

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show that $i_!(E) \in \mathrm{Ker}(K^{9-p}(S) \to K^{9-p}(S^{8-p}))$ and that

$$\{\mathrm{PD}_S[q \cdot WY_p]\}_{E_\infty^{9-p,0}} = [i_!(E)].$$

Thus, we must first use AHSS to detect possible anomalies, and then we can use the Gysin map to get the charge of a non-anomalous brane: such a charge belongs to the equivalence class reached by AHSS, so that the Gysin map gives more detailed information. For further remarks about this, we refer to the conclusions.

Moreover, we compare this picture with the case of rational coefficients. It is known that the Chern character provides isomorphisms $K(S) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H^{\mathrm{ev}}(S, \mathbb{Q})$ and $K^1(S) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H^{\mathrm{odd}}(S, \mathbb{Q})$, and that AHSS with rational coefficients degenerates at the second step, i.e. at the level of cohomology. Therefore, we gain a complete equivalence between the two K-theoretical approaches, being both equivalent to the old cohomological classification.

The paper is organized as follows. In Section 2 we discuss in detail the physical context underlying the K-theory classification of D-branes. In Sections 3 and 4 we introduce the topological tools needed to formulate our result, which is stated and proven in Section 5. In Section 6 we draw our conclusions.

2. Physical motivations

For simplicity we assume the ten-dimensional space–time S to be a compact manifold, so that also the D-brane world volumes are compact. This seems not physically reasonable, but it has more meaning if we suppose to have performed the Wick rotation in space–time, so that we work in a Euclidean setting. In this setting, we loose the physical interpretation of the D-brane world volume as a volume moving in time and of the charge q (actually all the homology class $[q \cdot Y_{p,t}]$ for $Y_{p,t}$ the restriction of the world volume at an instant t in a fixed reference frame) as a charge conserved in time. Thus, rather than considering the homology class of the D-brane volume at every instant of time, we prefer to consider the homology class of the entire world volume in S, using standard homology with compact support.

2.1. Classification

For a Dp-brane with $(p+1)$-dimensional world-volume WY_p and charge q, we consider the corresponding homology class in S:

$$[q \cdot WY_p] \in H_{p+1}(S, \mathbb{Z}) = \frac{Z_{p+1}(S, \mathbb{Z})}{B_{p+1}(S, \mathbb{Z})} = \mathbb{Z}^{b_{p+1}} \oplus_i \mathbb{Z}_{p_i^{n_i}} \tag{1}$$

where $Z_{p+1}(S, \mathbb{Z})$ denotes the group of singular $(p+1)$-cycles of S, $B_{p+1}(S, \mathbb{Z})$ the subgroup of $(p+1)$-boundaries, b_{p+1} the $(p+1)$th Betti number of S, and p_i is a prime number for every i. For what will follow, it is convenient to consider the cohomology of S rather than the homology. Hence, denoting by PD_S the Poincaré duality map on S,[1] we define the *charge density*:

$$\mathrm{PD}_S[q \cdot WY_p] \in H^{9-p}(S, \mathbb{Z}) = \frac{Z^{9-p}(S, \mathbb{Z})}{B^{9-p}(S, \mathbb{Z})} = \mathbb{Z}^{b_{p+1}} \oplus_i \mathbb{Z}_{p_i^{n_i}} \tag{2}$$

where $Z^{9-p}(S, \mathbb{Z})$ is the group of singular $(9-p)$-cocyles and $B^{9-p}(S, \mathbb{Z})$ the subgroup of $(9-p)$-coboundaries. This classification encounters some problems due to the presence of quantum anomalies. Two remarkable examples are as follows:

- a brane wrapping a cycle $WY_p \subset S$ is Freed–Witten anomalous if its third integral Stiefel–Whitney class $W_3(WY_p)$ is not zero, and hence, not all the cycles are allowed [6,1];
- given a world-volume WY_p with $W_3(WY_p) \neq 0$, it can be interpreted as an MMS-instanton in the Minkowskian setting [7,1]. In this case, there are cycles intersecting WY_p in $\mathrm{PD}_{WY_p}(W_3(WY_p))$ which, although homologically non-trivial in general, are actually unstable.

The two points above imply that

- the numerator $Z_{p+1}(S, \mathbb{Z})$ of (1) is too large since it contains anomalous cycles;
- the denominator $B_{p+1}(S, \mathbb{Z})$ of (1) is too small since it does not cut all the unstable charges.

There are other possible anomalies, although not yet completely understood, some of which are probably related to homology classes not representable by a smooth submanifold [8,9,1].

We start by considering the case of world volumes of *even* codimension in S, i.e. we start with IIB superstring theory. To solve the problems mentioned above, one possible tool seems to be the Atiyah–Hirzebruch spectral sequence [5]. Choosing a finite simplicial decomposition [10] of the space–time manifold S, and considering the filtration $S = S^{10} \supset \cdots \supset S^0$

[1] As we said above, we are assuming for simplicity that the space–time is a compact manifold (without singularities), and we also suppose it is orientable; thus, Poincaré duality holds.